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SPECTRAL ELEMENT DISCRETIZATION OF THE MAXWELL EQUATIONS

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ABSTRACT. We consider a variational problem which is equivalent to the electromagnetism system with absorbing conditions on a part of the boundary, and we prove that it is well-posed. Next we propose a discretization relying on a finite difference scheme for the time variable and on spectral elements for the space variables, and we derive error estimates between the exact and discrete solutions.

RÉSUMÉ. On considère un problème variationnel équivalent aux équations de l'électromagnétisme avec conditions aux limites absorbantes sur une partie de la frontière, qu'on prouve être bien posé. Puis on propose une discrétisation de ce problème par schéma aux différences finies en temps et éléments spectraux en espace, et on établit des estimations d'erreur entre solutions exacte et approchée.

1. INTRODUCTION

The aim of this paper is to analyze a spectral element discretization of the time-dependent Maxwell equations in a two- or three-dimensional bounded domain, when the boundary is made of two connected parts: the first one is absorbing, the second one is conducting. Indeed, much work has been done for the finite element discretization of such a problem, both in the standard version (see for instance [ADHRS], [Bo], [H] and the references therein) and in the p and $h-p$ versions (see [M1] and [M2]). But it seems that the spectral discretization has less been studied in this framework (see [ABG], [BG]).

In contrast to the finite element method, spectral techniques are known for their infinite accuracy, in the sense that the order of convergence is only limited by the regularity of the exact solution: this results from the approximation by high degree polynomials. And solving the Maxwell system with spectral accuracy is important in a number of applications, especially when it is coupled with other equations (Vlasov, Schrödinger, ...), in order to ensure that a poor approximation of the electromagnetic field does not pollute the other unknowns.

The drawback of the pure spectral method is that it allows one to work only in tensorized domains, *i.e.* rectangles and rectangular parallelepipeds. But the spectral element method allows one to handle more general geometries, more precisely, domains that admit a decomposition into disjoint tensorized subdomains. The extension to completely general types of geometry is currently used; it relies on transformations of these subdomains and will not be considered here, since it just adds too many technicalities to the proofs for similar results.

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In this paper, we first write the Maxwell equations in a decoupled second order travelling wave form. Indeed, a large number of forms and models exist for the Maxwell system, see [RS]; we choose this one for its generality. We present the variational formulation of this problem that we intend to discretize. Such a formulation involves a nontrivial subspace of functions with square-integrable curl, and its equivalence with the initial partial differential equations relies on a density result which is established in [BBCD].

The space discretization is obtained by applying a Galerkin method with numerical integration to the variational formulation. The time discretization is performed thanks to the Newmark and leap-frog schemes, as is standard for Maxwell's equations. Then, we prove that, if a Courant–Friedrichs–Lévy condition is satisfied, the full discrete problem has a unique solution, and it is stable in the sense that an appropriate norm of the discrete solution is bounded by quantities depending only on the data.

Thanks to this stability property and to the consistency of the time scheme, deriving error estimates only requires polynomial approximation properties in the variational space. We prove them by extending the finite element arguments of Nédélec [N1] to the spectral case. The final estimates are of spectral type with respect to the space discretization. So the method provides an accurate discretization of the Maxwell equations, which could still be improved by combining the spectral element discretization with a higher order time scheme.

An outline of the paper is as follows. In Section 2, we recall the Maxwell system, we write the equivalent variational formulation and we study its well-posedness. In Section 3, we present the space-discrete problem and the full discrete problem, and we prove the stability of the latter. Section 4 is devoted to polynomial approximation results, first in a cube, and second with domain decomposition. In Section 5, we prove the error estimates between the exact and the discrete solution.

2. The continuous problem

Let Ω denote a bounded domain in $\mathbb{R}^d$, $d = 2$ or 3, such that its boundary is made of two connected parts (see Figure 1):

- the exterior one Γ_a (a stands for absorbing) is rectangular,
- the interior one Γ_c (c stands for conducting) is Lipschitz-continuous.

We denote by $\boldsymbol{n}$ the unit outward normal vector to Ω on $\Gamma_a \cup \Gamma_c$.

The problem and its variational formulation. First, for the three-dimensional case, we consider the Maxwell–Ampère system in the domain Ω:

$$-\varepsilon\, \partial_t \boldsymbol{e} + \frac{1}{\mu}\, \mathbf{curl}\, \boldsymbol{b} = \boldsymbol{j}, \tag{2.1}$$

$$\operatorname{div} \boldsymbol{b} = 0, \tag{2.2}$$

the Faraday equation

$$\partial_t \boldsymbol{b} + \mathbf{curl}\, \boldsymbol{e} = \mathbf{0}, \tag{2.3}$$

and the Gauss law

$$\varepsilon \operatorname{div} \boldsymbol{e} = \rho, \tag{2.4}$$

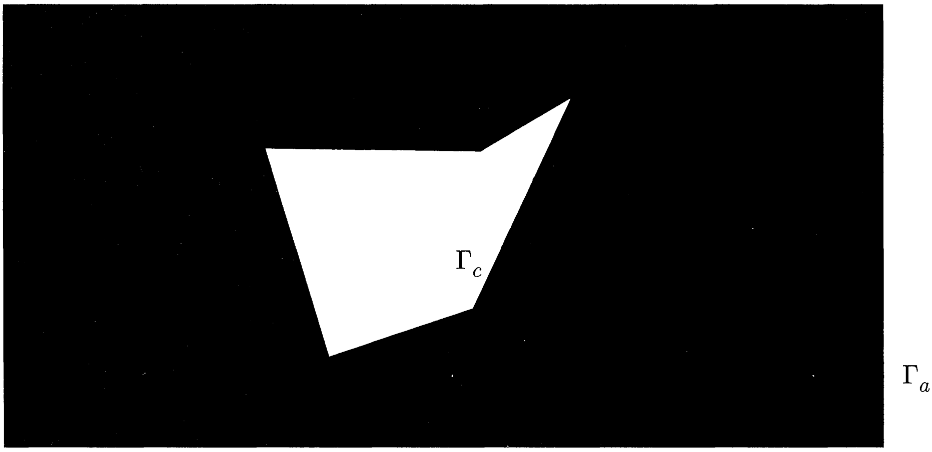

FIGURE 1

where the unknowns are the electric field $\boldsymbol{e}$ and the magnetic field $\boldsymbol{b}$. The dielectric permittivity ε and the magnetic permeability μ are assumed to be positive constants. The data are the current density vector $\boldsymbol{j}$ and the charge density ρ. These equations are provided with the perfectly conducting boundary conditions on Γ_c

$$\boldsymbol{e} \times \boldsymbol{n} = \boldsymbol{0}, \tag{2.5}$$

and with the Silver-Müller boundary conditions on Γ_a

$$(\boldsymbol{e} - \frac{1}{\sqrt{\mu\varepsilon}}\, \boldsymbol{b} \times \boldsymbol{n}) \times \boldsymbol{n} = \boldsymbol{0}. \tag{2.6}$$

In this paper, we are interested in solving the Cauchy problem, so we consider the following initial conditions in Ω:

$$\boldsymbol{e}_{|t=0} = \boldsymbol{e}_0 \quad \text{and} \quad \boldsymbol{b}_{|t=0} = \boldsymbol{b}_0. \tag{2.7}$$

The equations are the same in the two-dimensional case if all derivatives with respect to the third variable are taken equal to zero and with the convention

$$\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix}, \qquad \boldsymbol{e} = \begin{pmatrix} 0 \\ 0 \\ e \end{pmatrix}.$$

By differentiating equation (2.3) with respect to the time t and combining with the curl of equation (2.1), we derive

$$\varepsilon\, \partial_t^2 \boldsymbol{b} + \frac{1}{\mu}\, \mathbf{curl}\,(\mathbf{curl}\, \boldsymbol{b}) = \mathbf{curl}\, \boldsymbol{j}.$$

Similar transformations on boundary and initial conditions lead to the system

$$\begin{cases} \varepsilon\, \partial_t^2 \boldsymbol{b} + \frac{1}{\mu}\, \mathbf{curl}\,(\mathbf{curl}\, \boldsymbol{b}) = \mathbf{curl}\, \boldsymbol{j} & \text{in } \Omega, \\ (\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{n} = \mu\, (\boldsymbol{j} \times \boldsymbol{n}) & \text{on } \Gamma_c, \\ \big(-\sqrt{\mu\varepsilon}\, (\partial_t \boldsymbol{b} \times \boldsymbol{n}) + \mathbf{curl}\, \boldsymbol{b}\big) \times \boldsymbol{n} = \mu\, (\boldsymbol{j} \times \boldsymbol{n}) & \text{on } \Gamma_a, \\ \boldsymbol{b}_{|t=0} = \boldsymbol{b}_0 \quad \text{and} \quad (\partial_t \boldsymbol{b})_{|t=0} = -\mathbf{curl}\, \boldsymbol{e}_0 & \text{in } \Omega, \end{cases} \tag{2.8}$$

where the only unknown is now the function $\boldsymbol{b}$.

Conversely, let $\boldsymbol{b}$ be any solution of (2.8). Then, if $\boldsymbol{b}$ is smooth enough (*i.e.* if its **curl** belongs to $L^2(\Omega)^3$), the function $\boldsymbol{e}$ is uniquely defined by (2.1) and the initial condition:

$$\begin{cases} \varepsilon\, \partial_t \boldsymbol{e} = \frac{1}{\mu}\, \mathbf{curl}\, \boldsymbol{b} - \boldsymbol{j} & \text{in } \Omega, \\ \boldsymbol{e}_{|t=0} = \boldsymbol{e}_0 & \text{in } \Omega. \end{cases}$$

So, it is also readily checked that equation (2.3) holds. Next, with the assumption

$$\operatorname{div} \boldsymbol{b}_0 = 0 \quad \text{in } \Omega,$$

taking the divergence of (2.3) yields (2.2). Also, (2.4) is a consequence of the conservation law for the charge density

$$\begin{cases} \partial_t \rho + \operatorname{div} \boldsymbol{j} = 0 & \text{in } \Omega, \\ \rho_{|t=0} = \varepsilon \operatorname{div} \boldsymbol{e}_0 & \text{in } \Omega. \end{cases}$$

Finally, from the two assumptions

$$\boldsymbol{e}_0 \times \boldsymbol{n} = 0 \quad \text{on } \Gamma_c, \qquad (\boldsymbol{e}_0 - \frac{1}{\sqrt{\mu\varepsilon}}\, \boldsymbol{b}_0 \times \boldsymbol{n}) \times \boldsymbol{n} = 0 \quad \text{on } \Gamma_a,$$

we recover the boundary conditions (2.5) and (2.6). So, all the equations (2.1)–(2.7) are satisfied.

Consequently, from now on, we only consider the system (2.8). Its analysis is well-known, see for instance [RS, §3]; however we prefer to recall it in view of the analogous arguments that will be needed for the discrete problem.

Let us recall that the space $H(\mathbf{curl}, \Omega)$ is the space of functions in $L^2(\Omega)^3$ whose **curl**s belong to $L^2(\Omega)^3$. It is provided with the norm

$$\|\boldsymbol{v}\|_{H(\mathbf{curl},\Omega)} = \left(\|\boldsymbol{v}\|^2_{L^2(\Omega)^3} + \|\mathbf{curl}\, \boldsymbol{v}\|^2_{L^2(\Omega)^3} \right)^{\frac{1}{2}}.$$

We introduce its subspace $V(\Omega)$ of all functions in $H(\mathbf{curl}, \Omega)$ whose tangential traces on Γ_a belong to $L^2(\Gamma_a)^2$, provided with the norm

$$\|\boldsymbol{v}\|_{V(\Omega)} = \left(\|\boldsymbol{v}\|^2_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{v} \times \boldsymbol{n}\|^2_{L^2(\Gamma_a)^2} \right)^{\frac{1}{2}}.$$

The density of $\mathcal{D}(\overline{\Omega})$ in $H(\mathbf{curl}, \Omega)$ is well-known, see [GR, Chap. I, Thm. 2.4]. However it does not yield the density of $\mathcal{D}(\overline{\Omega})$ in $V(\Omega)$. We refer to [BBCD] for this last result.

We are now in a position to write the equivalent variational formulation of system (2.8). It reads: *find* $\boldsymbol{b}$ *in* $V(\Omega)$ *such that*

$$\begin{aligned} (2.9) \qquad \forall \boldsymbol{v} \in V(\Omega), \quad & \varepsilon \int_\Omega (\partial_t^2 \boldsymbol{b}) \,.\, \boldsymbol{v}\, d\boldsymbol{x} + \frac{1}{\mu} \int_\Omega \mathbf{curl}\, \boldsymbol{b} \,.\, \mathbf{curl}\, \boldsymbol{v}\, d\boldsymbol{x} \\ & + \sqrt{\frac{\varepsilon}{\mu}} \int_{\Gamma_a} \big((\partial_t \boldsymbol{b}) \times \boldsymbol{n}\big) \,.\, (\boldsymbol{v} \times \boldsymbol{n})\, d\boldsymbol{\tau} = \int_\Omega \boldsymbol{j} \,.\, \mathbf{curl}\, \boldsymbol{v}\, d\boldsymbol{x}. \end{aligned}$$

Of course, the initial conditions must be added:

$$(2.10) \qquad \boldsymbol{b}_{|t=0} = \boldsymbol{b}_0 \quad \text{and} \quad (\partial_t \boldsymbol{b})_{|t=0} = -\mathbf{curl}\, \boldsymbol{e}_0 \qquad \text{in } \Omega.$$

Proposition 2.1. *Let T be a positive real number. Assume that the function $\boldsymbol{j}$ belongs to $L^2(0, T; L^2(\Omega)^3)$ and that the initial data $\boldsymbol{b}_0$ and $\boldsymbol{e}_0$ belong to $V(\Omega)$ and $H(\mathbf{curl}, \Omega)$, respectively. Then, any solution $\boldsymbol{b}$ of* (2.8) *in $\mathcal{C}^0(0, T; V(\Omega)) \cap \mathcal{C}^1(0, T; L^2(\Omega)^3)$ is a solution of* (2.9) *in the distribution sense on $(0, T)$, and of*

(2.10). *Conversely, any solution of* (2.9) *satisfies the first line of* (2.8) *in a distribution sense, the second line in* $H^{-\frac{1}{2}}(\Gamma_c)^2$ *and the third line in* $L^2(\Gamma_a)^2$.

Proof. The first part of the proposition follows from the already quoted density result. The second part is derived by letting $\boldsymbol{v}$ in (2.9) run first through $\mathcal{D}(\Omega)^3$, second through the subspace of functions in $H^1(\Omega)^3$ vanishing on Γ_a and third through the subspace of functions in $V(\Omega)$ vanishing on Γ_c. □

We now intend to prove that problem (2.9)(2.10) is well-posed. However, the arguments are quite technical.

A priori estimate. This key estimate requires some regularity of the data.

Proposition 2.2. *Let* T *be a positive real number. Assume that the data* $\boldsymbol{b}_0$ *and* $\boldsymbol{e}_0$ *belong to* $H(\mathbf{curl},\Omega)$, *and that the function* $\boldsymbol{j}$ *belongs to* $H^1(0,T;L^2(\Omega)^3)$. *We set*

$$\kappa(\boldsymbol{j}) = 16\mu\Big(t\int_0^t \|(\partial_t\boldsymbol{j})(s)\|^2_{L^2(\Omega)^3}\,ds + \sup_{0\le s\le t}\|\boldsymbol{j}(s)\|^2_{L^2(\Omega)^3}\Big). \tag{2.11}$$

Then, any sufficiently smooth solution of problem (2.9)(2.10) *satisfies for all* t, $0<t\le T$,

(2.12)

$$\varepsilon\,\|(\partial_t\boldsymbol{b})(t)\|^2_{L^2(\Omega)^3} + \frac{1}{\mu}\,\|(\mathbf{curl}\,\boldsymbol{b})(t)\|^2_{L^2(\Omega)^3} + 2\sqrt{\frac{\varepsilon}{\mu}}\int_0^t \|(\partial_t\boldsymbol{b}\times\boldsymbol{n})(s)\|^2_{L^2(\Gamma_a)^2}\,ds$$
$$\le \kappa(\boldsymbol{j}) + 2\big(\frac{3}{\mu}\,\|\mathbf{curl}\,b_0\|^2_{L^2(\Omega)^3} + 2\varepsilon\,\|\mathbf{curl}\,e_0\|^2_{L^2(\Omega)^3}\big).$$

Proof. Taking $\boldsymbol{v}=\partial_t\boldsymbol{b}$ in (2.9) and integrating with respect to t yield

$$\frac{\varepsilon}{2}\,\|(\partial_t\boldsymbol{b})(t)\|^2_{L^2(\Omega)^3} + \frac{1}{2\mu}\,\|(\mathbf{curl}\,\boldsymbol{b})(t)\|^2_{L^2(\Omega)^3} + \sqrt{\frac{\varepsilon}{\mu}}\int_0^t \|(\partial_t\boldsymbol{b}\times\boldsymbol{n})(s)\|^2_{L^2(\Gamma_a)^2}\,ds$$
$$\le \int_0^t\int_\Omega \boldsymbol{j}\cdot\mathbf{curl}\,(\partial_t\boldsymbol{b})\,d\boldsymbol{x}\,ds + \frac{\varepsilon}{2}\,\|\mathbf{curl}\,\boldsymbol{e}_0\|^2_{L^2(\Omega)^3} + \frac{1}{2\mu}\,\|\mathbf{curl}\,\boldsymbol{b}_0\|^2_{L^2(\Omega)^3}.$$

Next, we integrate by parts with respect to t the first term in the right-hand side: for any positive constant γ,

$$\int_0^t\int_\Omega \boldsymbol{j}\cdot\mathbf{curl}\,(\partial_t\boldsymbol{b})\,d\boldsymbol{x}\,ds$$
$$\le \frac{1}{4\mu}\int_0^t \gamma\|(\mathbf{curl}\,\boldsymbol{b})(s)\|^2_{L^2(\Omega)^3}\,ds + \mu\int_0^t \frac{1}{\gamma}\|(\partial_t\boldsymbol{j})(s)\|^2_{L^2(\Omega)^3}\,ds$$
$$+\frac{1}{4\mu}\|(\mathbf{curl}\,\boldsymbol{b})(t)\|^2_{L^2(\Omega)^3} + \mu\,\|\boldsymbol{j}(t)\|^2_{L^2(\Omega)^3}$$
$$+\frac{1}{4\mu}\|\mathbf{curl}\,\boldsymbol{b}_0\|^2_{L^2(\Omega)^3} + \mu\|\boldsymbol{j}_{|t=0}\|^2_{L^2(\Omega)^3}.$$

Inserting this in the previous inequality leads to

$$\|(\mathbf{curl}\,\boldsymbol{b})(t)\|^2_{L^2(\Omega)^3} \le \int_0^t \gamma\|(\mathbf{curl}\,\boldsymbol{b})(s)\|^2_{L^2(\Omega)^3}\,ds + 4\mu^2\int_0^t \frac{1}{\gamma}\|(\partial_t\boldsymbol{j})(s)\|^2_{L^2(\Omega)^3}\,ds$$
$$+4\mu^2\,\|\boldsymbol{j}(t)\|^2_{L^2(\Omega)^3} + 4\mu^2\|\boldsymbol{j}_{|t=0}\|^2_{L^2(\Omega)^3}$$
$$+2\varepsilon\mu\,\|\mathbf{curl}\,\boldsymbol{e}_0\|^2_{L^2(\Omega)^3} + 3\,\|\mathbf{curl}\,\boldsymbol{b}_0\|^2_{L^2(\Omega)^3}.$$

So, denoting by $D(t^*)$ the maximum of $\|(\mathbf{curl}\,\boldsymbol{b})(s)\|^2_{L^2(\Omega)^3}$ for $0 \le s \le t^*$ and taking $\gamma = 1/2t^*$ yields

$$\frac{1}{2}D(t^*) \le 8\mu^2 t^* \int_0^{t^*} \|(\partial_t \boldsymbol{j})(s)\|^2_{L^2(\Omega)^3}\,ds \\ + 8\mu^2 \sup_{0\le s\le t^*} \|\boldsymbol{j}(s)\|^2_{L^2(\Omega)^3} + 2\varepsilon\mu\|\mathbf{curl}\,\boldsymbol{e}_0\|^2_{L^2(\Omega)^3} + 3\|\mathbf{curl}\,\boldsymbol{b}_0\|^2_{L^2(\Omega)^3}.$$

This gives the estimate for $D(t^*)$, and the other ones follow. $\square$

Existence and uniqueness results. Problem (2.9)(2.10) is very similar to one of the problems in Lions and Magenes [LM, Chap. III, (8.3)(8.4)]. However, the proof of [LM, Chap. III, Th. 8.1] would only be valid in our case if $\int_\Omega \boldsymbol{j}\,.\,\mathbf{curl}\,\boldsymbol{v}\,d\boldsymbol{x}$ were replaced by $\int_\Omega \mathbf{curl}\,\boldsymbol{j}\,.\,\boldsymbol{v}\,d\boldsymbol{x}$. So we now adapt this proof to problem (2.9)(2.10).

Theorem 2.3. *Let T be a positive real number. Assume that the data $\boldsymbol{b}_0$ and. $\boldsymbol{e}_0$ belong to $V(\Omega)$ and that the function $\boldsymbol{j}$ belongs to $H^1(0,T;L^2(\Omega)^3)$. Then, problem* (2.9)(2.10) *has a unique solution $\boldsymbol{b}$ satisfying*

(2.13)
$$\boldsymbol{b} \in L^2(0,T;H(\mathbf{curl},\Omega)), \quad \partial_t\boldsymbol{b} \in L^2(0,T;L^2(\Omega)^3), \quad \partial_t\boldsymbol{b}\times\boldsymbol{n} \in L^2(0,T;L^2(\Gamma_a)^2).$$

Moreover, this solution is such that

$$\boldsymbol{b} \in \mathcal{C}^0(0,T;H(\mathbf{curl},\Omega)), \qquad \partial_t\boldsymbol{b} \in \mathcal{C}^0(0,T;L^2(\Omega)^3), \tag{2.14}$$

and satisfies (2.12).

Proof. Since $V(\Omega)$ is a subspace of $L^2(\Omega)^3$, it is separable. So we can introduce an ordered basis $(\boldsymbol{w}_m)_m$ of $V(\Omega)$ (in the sense that the linear combinations of the $\boldsymbol{w}_m$ are dense in $V(\Omega)$). We denote by V_m the subspace spanned by $\boldsymbol{w}_0$, ... and $\boldsymbol{w}_m$. Then, we look for a solution $\boldsymbol{b}_m$ in $L^2(0,T;V_m)$ of the problem:

$$\forall \boldsymbol{v}_m \in V_m, \quad \varepsilon\int_\Omega (\partial_t^2\boldsymbol{b}_m)\,.\,\boldsymbol{v}_m\,d\boldsymbol{x} + \frac{1}{\mu}\int_\Omega \mathbf{curl}\,\boldsymbol{b}_m\,.\,\mathbf{curl}\,\boldsymbol{v}_m\,d\boldsymbol{x} \\ + \sqrt{\frac{\varepsilon}{\mu}}\int_{\Gamma_a}\big((\partial_t\boldsymbol{b}_m)\times\boldsymbol{n}\big)\,.\,(\boldsymbol{v}_m\times\boldsymbol{n})\,d\boldsymbol{\tau} = \int_\Omega \boldsymbol{j}\,.\,\mathbf{curl}\,\boldsymbol{v}_m\,d\boldsymbol{x}, \tag{2.15}$$

with the initial conditions

$$\boldsymbol{b}_{m\,|t=0} = \pi_m\boldsymbol{b}_0 \quad \text{and} \quad (\partial_t\boldsymbol{b}_m)_{|t=0} = -\mathbf{curl}\,(\pi_m\boldsymbol{e}_0) \qquad \text{in } \Omega \tag{2.16}$$

(π_m denotes the orthogonal projection operator from $V(\Omega)$ onto V_m). Writing $\boldsymbol{b}_m(t) = \sum_{\ell=0}^m \beta_\ell(t)\,\boldsymbol{w}_\ell$ and denoting by B the vector with components β_ℓ, we observe that the previous problem (2.15)(2.16) is equivalent to a linear system of ordinary differential equations of the type

$$\begin{cases} B'' + \mathcal{A}B + \mathcal{C}B' = J, \\ B_{|t=0} = B_0 \quad \text{and} \quad B'_{|t=0} = B_1. \end{cases}$$

So it has a unique solution $\boldsymbol{b}_m$. By the same arguments as in the proof of the previous Proposition 2.2, this function $\boldsymbol{b}_m$ also satisfies (2.12) and, in particular, there is a constant c, depending only on the data, such that

$$\|\boldsymbol{b}_m\|_{L^2(0,T;H(\mathbf{curl},\Omega))} + \|\partial_t\boldsymbol{b}_m\|_{L^2(0,T;L^2(\Omega)^3)} + \|\partial_t\boldsymbol{b}_m\times\boldsymbol{n}\|_{L^2(0,T;L^2(\Gamma_a)^2)} \le c.$$

Consequently, there exists a subsequence, still denoted by $(\boldsymbol{b}_m)_m$, converging to a function $\boldsymbol{b}$ weakly in $L^2(0,T;H(\mathbf{curl},\Omega))$ such that $(\partial_t\boldsymbol{b}_m)_m$ also converges to $\partial_t\boldsymbol{b}$

weakly in $L^2(0,T;L^2(\Omega)^3)$ and $(\partial_t \boldsymbol{b}_m \times \boldsymbol{n})_m$ also converges to $\partial_t \boldsymbol{b} \times \boldsymbol{n}$ weakly in $L^2(0,T;L^2(\Gamma_a)^2)$. Passing to the limit in (2.15)(2.16) now yields that $\boldsymbol{b}$ is a solution of (2.9)(2.10). Its uniqueness and further regularity easily follow from (2.12). □

The regularity properties of problem (2.9)(2.10) can be derived from Costabel and Dauge [Co][CD] (see also Moussaoui [Mo] for two-dimensional results). They are closely linked to the regularity of the Laplace, Dirichlet and Neumann equations in the same domain.

3. The discrete problem

We first make precise the geometry in which we work, and present the space discretization that relies on the standard spectral element method. Next we write the full discrete problem, we check that it is well-posed, and we give a stability estimate.

In what follows, we assume that Ω is a polyhedral domain in $\mathbb{R}^3$ such that there exists a finite number of (open) rectangular parallelepipeds Ω_k, $1 \le k \le K$, with edges parallel to the coordinate axes, satisfying

$$\overline{\Omega} = \bigcup_{k=1}^{K} \overline{\Omega}_k \quad \text{and} \quad \Omega_k \cap \Omega_{k'} = \emptyset, \quad 1 \le k < k' \le K. \tag{3.1}$$

As is well-known in the spectral element technique, this is fairly representative for general geometry since transforming rectangular parallelepipeds onto more general subdomains by a regular mapping adds only technical, not theoretical difficulties to the analysis. We make the following assumption on this decomposition.

Assumption 3.1. *The intersection of $\overline{\Omega}_k$ and $\overline{\Omega}_{k'}$, $1 \le k < k' \le K$, if not empty, is either a corner or an edge or a face of both Ω_k and $\Omega_{k'}$.*

Space discretization. Now, for any nonnegative integer n, we introduce the space $\mathbb{P}_n(\Omega_k)$ of polynomials with degree $\le n$ with respect to each variable. Then we fix a positive integer N and we define the discrete space

$$V_N(\Omega) = \Big\{ \boldsymbol{v}_N \in H(\mathbf{curl}, \Omega);\ \boldsymbol{v}_{N\,|\Omega_k} \in \mathbb{P}_N(\Omega_k)^3, 1 \le k \le K \Big\}. \tag{3.2}$$

It is readily checked that this is a subspace of $V(\Omega)$, so the discretization is conforming (we recall that such a discrete space is proposed by Nédélec [N2, Def. 6] in the finite element framework).

We start from the Gauss–Lobatto formula on $]-1,1[$: if $\xi_0 = -1$ and $\xi_N = 1$, there exist $N-1$ nodes ξ_j, $1 \le j \le N-1$ (which are the zeros of the first derivative of the Legendre polynomial of degree N) and $N+1$ weights ρ_j, $0 \le j \le N$, such that the following equality holds for all polynomials Φ of degree $\le 2N-1$:

$$\int_{-1}^{1} \Phi(\zeta)\, d\zeta = \sum_{j=0}^{N} \Phi(\xi_j)\, \rho_j. \tag{3.3}$$

By translation and homothety, these nodes are mapped onto Ω_k in each direction; we denote them by x_j^k, y_j^k and z_j^k, and the corresponding weights by $\rho_j^{x,k}$, $\rho_j^{y,k}$ and $\rho_j^{z,k}$. This allows us to define the discrete product on all functions u and v

continuous on each $\overline{\Omega}_k$:

$$((u,v))_N = \sum_{k=1}^{K} \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{\ell=0}^{N} u(x_i^k, y_j^k, z_\ell^k)\, v(x_i^k, y_j^k, z_\ell^k)\, \rho_i^{x,k}\rho_j^{y,k}\rho_\ell^{z,k}. \tag{3.4}$$

Similarly, let Γ_m, $1 \le m \le M$, be the faces of the Ω_k that are contained in Γ_a. Denoting by σ_i^m and τ_j^m, $1 \le m \le M$, the tangential coordinates of the nodes that belong to $\overline{\Gamma}_m$, and by $\rho_i^{\sigma,m}$ and $\rho_j^{\tau,m}$ the corresponding weights, we define the discrete product on functions continuous on each face $\overline{\Gamma}_m$:

$$((u,v))_{N,\Gamma_a} = \sum_{m=1}^{M} \sum_{i=0}^{N}\sum_{j=0}^{N} u(\sigma_i^m, \tau_j^m)\, v(\sigma_i^m, \tau_j^m)\, \rho_i^{\sigma,m}\rho_j^{\tau,m}. \tag{3.5}$$

We also define the associated seminorms

$$\|u\|_N = ((u,u))_N^{\frac{1}{2}}, \quad \|u\|_{N,\Gamma_a} = ((u,u))_{N,\Gamma_a}^{\frac{1}{2}}. \tag{3.6}$$

We denote by $\mathcal{I}_N^k$ the Lagrange interpolation operator (with values in $\mathbb{P}_N(\Omega_k)$) on the grid

$$\Xi_N^k = \Big\{(x_i^k, y_j^k, z_\ell^k),\ 0 \le i,j,\ell \le N\Big\}, \tag{3.7}$$

and by $\mathcal{I}_N$ the Lagrange interpolation operator on $\bigcup_{k=1}^K \Xi_N^k$.

This allows us to write the semidiscrete problem, for a.e. t in $]0,T]$: *find $\boldsymbol{b}_N$ in $V_N(\Omega)$ such that*

$$\begin{aligned}\forall \boldsymbol{v}_N \in V_N(\Omega),\quad & \varepsilon\,((\partial_t^2 \boldsymbol{b}_N, \boldsymbol{v}_N))_N + \frac{1}{\mu}\,((\mathbf{curl}\,\boldsymbol{b}_N, \mathbf{curl}\,\boldsymbol{v}_N))_N \\ & + \sqrt{\frac{\varepsilon}{\mu}}((\partial_t \boldsymbol{b}_N \times \boldsymbol{n}, \boldsymbol{v}_N \times \boldsymbol{n}))_{N,\Gamma_a} = ((\boldsymbol{j}, \mathbf{curl}\,\boldsymbol{v}_N))_N.\end{aligned} \tag{3.8}$$

We must also enforce the initial conditions:

$$\boldsymbol{b}_{N\,|t=0} = \boldsymbol{b}_{0N} \quad\text{and}\quad (\partial_t \boldsymbol{b}_N)_{|t=0} = -\mathbf{curl}\,\boldsymbol{e}_{0N} \qquad \text{in } \Omega, \tag{3.9}$$

where $\boldsymbol{b}_{0N}$ and $\boldsymbol{e}_{0N}$ are approximations of $\boldsymbol{b}_0$ and $\boldsymbol{e}_0$ in $V_N(\Omega)$.

Let us recall [BM, (13.20)] the positivity property, which holds for any polynomial φ_N with degree $\le N$:

$$\|\varphi_N\|^2_{L^2(-1,1)} \le \sum_{j=0}^{N} \varphi_N^2(\xi_j)\,\rho_j \le 3\,\|\varphi_N\|^2_{L^2(-1,1)}. \tag{3.10}$$

By the same arguments as in the proof of Proposition 2.2 combined with the previous positivity property of the discrete product, it is readily checked that there exists a constant c independent of N such that the solution $\boldsymbol{b}_N$ of (3.8)(3.9) satisfies

$$\begin{aligned}& \varepsilon\,\|(\partial_t \boldsymbol{b}_N)(t)\|^2_{L^2(\Omega)^3} + \frac{1}{\mu}\,\|(\mathbf{curl}\,\boldsymbol{b}_N)(t)\|^2_{L^2(\Omega)^3} \\ & \quad + 2\sqrt{\frac{\varepsilon}{\mu}} \int_0^t \|(\partial_t \boldsymbol{b}_N \times \boldsymbol{n})(s)\|^2_{L^2(\Gamma_a)^2}\,ds \\ & \le c\,\Big(\kappa(\mathcal{I}_N \boldsymbol{j}) + \frac{3}{\mu}\,\|\mathbf{curl}\,b_{0N}\|^2_{L^2(\Omega)^3} + 2\varepsilon\,\|\mathbf{curl}\,e_{0N}\|^2_{L^2(\Omega)^3}\Big).\end{aligned} \tag{3.11}$$

So, problem (3.8)(3.9) has a unique solution in $\mathcal{C}^1(0,T; V_N(\Omega))$.

The fully discrete problem. Let δt denote the time step (we assume that $\delta t \le 1$ and that $T = P\delta t$). The idea is to define a sequence $(\boldsymbol{b}_N^p)_{0\le p\le P}$ such that $\boldsymbol{b}_N^p$ approximates the solution $\boldsymbol{b}$ of problem (2.9)(2.10) at times $p\delta t$. We first take as initial conditions

$$\boldsymbol{b}_N^0 = \boldsymbol{b}_{0N} \quad \text{and} \quad \boldsymbol{b}_N^1 = \boldsymbol{b}_{0N} - \delta t\, \mathbf{curl}\, \boldsymbol{e}_{0N} \qquad \text{in } \Omega. \tag{3.12}$$

Next, we use a Newmark scheme to discretize the second derivative with respect to time, and a leap-frog scheme to discretize the first derivative. This leads to the problem, for $1 \le p \le P-1$: *find* $\boldsymbol{b}_N^{p+1}$ *in* $V_N(\Omega)$ *such that*

$$\begin{aligned}\forall \boldsymbol{v}_N \in V_N(\Omega),\quad & \varepsilon\,((\frac{\boldsymbol{b}_N^{p+1} - 2\boldsymbol{b}_N^p + \boldsymbol{b}_N^{p-1}}{(\delta t)^2}, \boldsymbol{v}_N))_N + \frac{1}{\mu}\,((\mathbf{curl}\, \boldsymbol{b}_N^p, \mathbf{curl}\, \boldsymbol{v}_N))_N \\ & + \sqrt{\frac{\varepsilon}{\mu}}((\frac{\boldsymbol{b}_N^{p+1} - \boldsymbol{b}_N^{p-1}}{2\delta t} \times \boldsymbol{n}, \boldsymbol{v}_N \times \boldsymbol{n}))_{N,\Gamma_a} = ((\boldsymbol{j}(p\delta t), \mathbf{curl}\, \boldsymbol{v}_N))_N.\end{aligned} \tag{3.13}$$

This last equation can equivalently be written as

$$\varepsilon\,((\boldsymbol{b}_N^{p+1}, \boldsymbol{v}_N))_N + \frac{\delta t}{2}\sqrt{\frac{\varepsilon}{\mu}}((\boldsymbol{b}_N^{p+1} \times \boldsymbol{n}, \boldsymbol{v}_N \times \boldsymbol{n}))_{N,\Gamma_a} = ((\boldsymbol{f}_N^p, \boldsymbol{v}_N))_N,$$

where $\boldsymbol{f}_N^p$ is known by the induction hypothesis. So, $\boldsymbol{b}_N^{p+1}$ is the solution of a finite dimensional square linear system, and, from the positivity property (3.10), equations (3.12)(3.13) uniquely define the sequence $(\boldsymbol{b}_N^p)_{0\le p\le P}$ for any function $\boldsymbol{j}$ in $\mathcal{C}^0([0,T]\times\overline{\Omega})$.

The scheme that we use is explicit with respect to the term $((\mathbf{curl}\, \boldsymbol{b}_N^p, \mathbf{curl}\, \boldsymbol{v}_N))_N$, so a Courant–Friedrichs–Lévy condition is necessary to ensure the stability of the problem. However, working with the Maxwell equations most often requires very small time steps, so that this condition is not restrictive in practical situations.

A priori estimates. The arguments for proving the stability estimate are the discrete analogues of those in the continuous case — see the proof of Proposition 2.2. They make use of polynomial inverse inequalities that are well-known [BM, §5], but we choose to give a fine version of them in order to optimize the Courant–Friedrichs–Lévy condition.

Lemma 3.2. *Any polynomial* $\boldsymbol{v}_N$ *in* $V_N(\Omega)$ *satisfies the following inverse inequality:*

$$\|\mathbf{curl}\, \boldsymbol{v}_N\|_N \le \rho_N(\Omega)\, \|\boldsymbol{v}_N\|_N, \tag{3.14}$$

where $\rho_N(\Omega)$ *is given by*

$$\rho_N(\Omega) = \Big(\frac{N(N+1)}{\pi} + \frac{\pi}{4}\Big) \sup_{1\le k\le K} \sigma(k), \tag{3.15}$$

and $\sigma(k)$ *is the ratio of the area of the boundary of* Ω_k *to its volume.*

Proof. First, on each Ω_k, it is proved in [BG, §5] that any polynomial $\boldsymbol{v}_N$ in $\mathbb{P}_N(\Omega_k)^3$ satisfies

$$\begin{aligned}&\sum_{i=0}^N\sum_{j=0}^N\sum_{\ell=0}^N (\mathbf{curl}\, \boldsymbol{v}_N)^2(x_i^k, y_j^k, z_\ell^k)\, \rho_i^{x,k}\, \rho_j^{y,k}\, \rho_\ell^{z,k} \\ &\qquad \le \sigma(k)^2\, c_N^2 \sum_{i=0}^N\sum_{j=0}^N\sum_{\ell=0}^N \boldsymbol{v}_N^2(x_i^k, y_j^k, z_\ell^k)\, \rho_i^{x,k}\, \rho_j^{y,k}\, \rho_\ell^{z,k},\end{aligned}$$

where c_N is the smallest possible constant in the inverse inequlity

$$\forall \varphi_N \in \mathbb{P}_N(-1,1), \quad \|\varphi_N'\|_{L^2(-1,1)} \leq c_N \, \|\varphi_N\|_{L^2(-1,1)}.$$

Equivalently, it is the largest eigenvalue λ of the discrete Neumann problem: find φ_N in $\mathbb{P}_N(-1,1)$ such that

$$\forall \chi_N \in \mathbb{P}_N(-1,1), \quad \int_{-1}^{1} \varphi_N'(\zeta)\chi_N'(\zeta)\, d\zeta = \lambda^2 \int_{-1}^{1} \varphi_N(\zeta)\chi_N(\zeta)\, d\zeta.$$

Such a constant is evaluated in [AB, Remark 4.3]:

$$c_N = \frac{N(N+1)}{\pi} + \frac{\pi}{12} + \frac{\pi}{2}\,\big(1 + \frac{\pi^2}{45}\big)\, \frac{1}{N(N+1)} + \mathcal{O}(N^{-4}) \leq \frac{N(N+1)}{\pi} + \frac{\pi}{4}.$$

$\square$

Proposition 3.3. *Let T be a positive real number. Assume that the function $\boldsymbol{j}$ belongs to $\mathcal{C}^0(]0,T] \times \overline{\Omega})$. We set*

$$\kappa_N^p(\boldsymbol{j}) = 9\Big(\mu\delta t T \sum_{q=1}^{p-1} \|\frac{(\mathcal{I}_N\boldsymbol{j})\big((q+1)\delta t\big) - (\mathcal{I}_N\boldsymbol{j})\big((q-1)\delta t\big)}{\delta t}\|^2_{L^2(\Omega)^3} + \frac{21\mu}{2} \sup_{0\leq q\leq p} \|(\mathcal{I}_N\boldsymbol{j})(q\delta t)\|^2_{L^2(\Omega)^3}\Big). \tag{3.16}$$

Then, if the parameters N and δt satisfy the inequality

$$\delta t\, \rho_N(\Omega) \leq \sqrt{\varepsilon\mu}, \tag{3.17}$$

any solution of equations (3.12)(3.13) *satisfies for all p, $1 \leq p \leq P-1$,*

$$\frac{\varepsilon}{8}\,\|\frac{\boldsymbol{b}_N^{p+1} - \boldsymbol{b}_N^p}{\delta t}\|^2_{L^2(\Omega)^3} + \frac{1}{4\mu}\,\|\mathbf{curl}\,\boldsymbol{b}_N^p\|^2_{L^2(\Omega)^3} + 2\delta t\sqrt{\frac{\varepsilon}{\mu}} \sum_{q=1}^{p} \|\frac{\boldsymbol{b}_N^{q+1} - \boldsymbol{b}_N^q}{\delta t} \times \boldsymbol{n}\|^2_{L^2(\Gamma_a)^2} \leq \kappa_N^p(\boldsymbol{j}) + 9\big(\frac{7}{2\mu}\,\|\mathbf{curl}\, b_{0N}\|^2_{L^2(\Omega)^3} + 5\varepsilon\,\|\mathbf{curl}\, e_{0N}\|^2_{L^2(\Omega)^3}\big). \tag{3.18}$$

Proof. The idea is to take $\boldsymbol{v}_N$ equal to $\boldsymbol{b}_N^{p+1} - \boldsymbol{b}_N^{p-1}$ in (3.13). This yields

$$\varepsilon\,\|\frac{\boldsymbol{b}_N^{p+1} - \boldsymbol{b}_N^p}{\delta t}\|^2_N - \varepsilon\,\|\frac{\boldsymbol{b}_N^p - \boldsymbol{b}_N^{p-1}}{\delta t}\|^2_N + \frac{1}{\mu}\,\|\mathbf{curl}\,\boldsymbol{b}_N^{p+1}\|^2_N - \frac{1}{\mu}\,\|\mathbf{curl}\,\boldsymbol{b}_N^p\|^2_N$$
$$- \frac{1}{\mu}\,((\mathbf{curl}\,(\boldsymbol{b}_N^{p+1} - \boldsymbol{b}^p), \mathbf{curl}\,\boldsymbol{b}_N^{p+1}))_N + \frac{1}{\mu}\,((\mathbf{curl}\,(\boldsymbol{b}_N^p - \boldsymbol{b}_N^{p-1}), \mathbf{curl}\,\boldsymbol{b}_N^p))_N$$
$$+ 2\delta t\sqrt{\frac{\varepsilon}{\mu}}\|\frac{\boldsymbol{b}_N^{p+1} - \boldsymbol{b}_N^{p-1}}{2\delta t} \times \boldsymbol{n}\|^2_{N,\Gamma_a} \leq ((\boldsymbol{j}(p\delta t), \mathbf{curl}\,(\boldsymbol{b}_N^{p+1} - \boldsymbol{b}_N^{p-1}))_N.$$

Next we sum over p:

$$
\begin{aligned}
&\varepsilon \,\| \frac{\boldsymbol{b}_N^{p+1} - \boldsymbol{b}_N^p}{\delta t} \|_N^2 + \frac{1}{\mu} \,\| \mathbf{curl}\, \boldsymbol{b}_N^{p+1} \|_N^2 + 2\delta t \sqrt{\frac{\varepsilon}{\mu}} \sum_{q=1}^{p} \| \frac{\boldsymbol{b}_N^{q+1} - \boldsymbol{b}_N^{q-1}}{2\delta t} \times \boldsymbol{n} \|_{N,\Gamma_a}^2 \\
&\quad \le \frac{3}{4\mu} \,\| \mathbf{curl}\, (\boldsymbol{b}_N^{p+1} - \boldsymbol{b}_N^p) \|_N^2 + \frac{1}{3\mu} \,\| \mathbf{curl}\, \boldsymbol{b}_N^{p+1} \|_N^2 \\
&\qquad + \sum_{q=1}^{p} ((\boldsymbol{j}(q\delta t), \mathbf{curl}\, (\boldsymbol{b}_N^{q+1} - \boldsymbol{b}_N^{q-1}))_N \\
&\qquad + \varepsilon \| \mathbf{curl}\, \boldsymbol{e}_{0N} \|_N^2 + \frac{1}{2\mu} \,\| \mathbf{curl}\, (\boldsymbol{b}_N^1 - \boldsymbol{b}_N^0) \|_N^2 + \frac{3}{2\mu} \,\| \mathbf{curl}\, \boldsymbol{b}_N^1 \|_N^2 .
\end{aligned} \tag{3.19}
$$

We have to compute

$$
\begin{aligned}
&\sum_{q=1}^{p} ((\boldsymbol{j}(q\delta t), \mathbf{curl}\, (\boldsymbol{b}_N^{q+1} - \boldsymbol{b}_N^{q-1}))_N \\
&\quad = -\delta t \sum_{q=1}^{p-1} ((\frac{\boldsymbol{j}\big((q+1)\delta t\big) - \boldsymbol{j}\big((q-1)\delta t\big)}{\delta t}, \mathbf{curl}\, \boldsymbol{b}_N^q))_N \\
&\qquad + ((\boldsymbol{j}\big(p\delta t\big) + \boldsymbol{j}\big((p-1)\delta t\big), \mathbf{curl}\, (\boldsymbol{b}_N^{p+1}))_N \\
&\qquad\qquad - ((\boldsymbol{j}\big((p-1)\delta t\big), \mathbf{curl}\, (\boldsymbol{b}_N^{p+1} - \boldsymbol{b}_N^p)))_N \\
&\qquad - ((\boldsymbol{j}\big(\delta t\big) + \boldsymbol{j}\big(0\big), \mathbf{curl}\, \boldsymbol{b}_N^0))_N - ((\boldsymbol{j}\big(0\big), \mathbf{curl}\, (\boldsymbol{b}_N^1 - \boldsymbol{b}_N^0)))_N ,
\end{aligned}
$$

whence, for any positive γ,

$$
\begin{aligned}
&\sum_{q=1}^{p} ((\boldsymbol{j}(q\delta t), \mathbf{curl}\, (\boldsymbol{b}_N^{q+1} - \boldsymbol{b}_N^{q-1}))_N \\
&\quad \le \frac{\delta t}{2\mu} \sum_{q=1}^{p-1} \gamma \| \mathbf{curl}\, \boldsymbol{b}_N^q \|_N^2 + \lambda_N^p(\boldsymbol{j}) + \frac{1}{6\mu} \,\| \mathbf{curl}\, \boldsymbol{b}_N^{p+1} \|_N^2 + \frac{1}{8\mu} \,\| \mathbf{curl}\, (\boldsymbol{b}_N^{p+1} - \boldsymbol{b}_N^p) \|_N^2 \\
&\qquad\qquad + \frac{1}{2\mu} \,\| \mathbf{curl}\, \boldsymbol{b}_N^0 \|_N^2 + \frac{1}{2\mu} \,\| \mathbf{curl}\, (\boldsymbol{b}_N^1 - \boldsymbol{b}_N^0) \|_N^2 ,
\end{aligned}
$$

with

$$
\begin{aligned}
\lambda_N^p(\boldsymbol{j}) = {} & \frac{\mu \delta t}{2} \sum_{q=1}^{p-1} \frac{1}{\gamma} \| \frac{(\mathcal{I}_N \boldsymbol{j})\big((q+1)\delta t\big) - (\mathcal{I}_N \boldsymbol{j})\big((q-1)\delta t\big)}{\delta t} \|_N^2 \\
& + 3\mu \,\| (\mathcal{I}_N \boldsymbol{j})\big(p\delta t\big) \|_N^2 + 5\mu \,\| (\mathcal{I}_N \boldsymbol{j})\big((p-1)\delta t\big) \|_N^2 \\
& + \mu \,\| (\mathcal{I}_N \boldsymbol{j})\big(\delta t\big) \|_N^2 + \frac{3\mu}{2} \,\| (\mathcal{I}_N \boldsymbol{j})\big(0\big) \|_N^2 .
\end{aligned}
$$

Inserting this last inequality in (3.19) yields

$$\varepsilon \,\|\frac{\boldsymbol{b}_N^{p+1}-\boldsymbol{b}_N^p}{\delta t}\|_N^2 + \frac{1}{2\mu}\,\|\mathbf{curl}\,\boldsymbol{b}_N^{p+1}\|_N^2 + 2\delta t\sqrt{\frac{\varepsilon}{\mu}}\sum_{q=1}^{p}\|\frac{\boldsymbol{b}_N^{q+1}-\boldsymbol{b}_N^{q-1}}{2\delta t}\times\boldsymbol{n}\|_{N,\Gamma_a}^2$$

$$\leq \frac{\delta t}{2\mu}\sum_{q=1}^{p-1}\gamma\|\mathbf{curl}\,\boldsymbol{b}_N^q\|_N^2 + \lambda_N^p(\boldsymbol{j}) + \frac{7}{8\mu}\,\|\mathbf{curl}\,(\boldsymbol{b}_N^{p+1}-\boldsymbol{b}_N^p)\|_N^2 \tag{3.20}$$

$$+\,\varepsilon\|\mathbf{curl}\,\boldsymbol{e}_{0N}\|_N^2 + \frac{4}{\mu}\,\|\mathbf{curl}\,(\boldsymbol{b}_N^1-\boldsymbol{b}_N^0)\|_N^2 + \frac{7}{2\mu}\,\|\mathbf{curl}\,\boldsymbol{b}_N^0\|_N^2.$$

We use the inverse inequality (3.14) and (3.17) to derive

$$\frac{7}{8\mu}\,\|\mathbf{curl}\,(\boldsymbol{b}_N^{p+1}-\boldsymbol{b}_N^p)\|_N^2 \leq \frac{7\rho_N(\Omega)^2}{8\mu}\,\|\boldsymbol{b}_N^{p+1}-\boldsymbol{b}_N^p\|_N^2 \leq \frac{7\varepsilon}{8}\,\|\frac{\boldsymbol{b}_N^{p+1}-\boldsymbol{b}_N^p}{\delta t}\|_N^2,$$

and the same inequality for bounding $\frac{4}{\mu}\,\|\mathbf{curl}\,(\boldsymbol{b}_N^1-\boldsymbol{b}_N^0)\|_N^2$. Since $\delta t\leq 1$, this gives

$$\frac{\varepsilon}{8}\,\|\frac{\boldsymbol{b}_N^{p+1}-\boldsymbol{b}_N^p}{\delta t}\|_N^2 + \frac{1}{2\mu}\,\|\mathbf{curl}\,\boldsymbol{b}_N^{p+1}\|_N^2 + 2\delta t\sqrt{\frac{\varepsilon}{\mu}}\sum_{q=1}^{p}\|\frac{\boldsymbol{b}_N^{q+1}-\boldsymbol{b}_N^{q-1}}{2\delta t}\times\boldsymbol{n}\|_{N,\Gamma_a}^2 \tag{3.21}$$

$$\leq \frac{\delta t}{2\mu}\sum_{q=0}^{p-1}\|\mathbf{curl}\,\boldsymbol{b}_N^q\|_N^2 + \lambda_N^p(\boldsymbol{j}) + 5\varepsilon\,\|\mathbf{curl}\,\boldsymbol{e}_{0N}\|_N^2 + \frac{7}{2\mu}\,\|\mathbf{curl}\,\boldsymbol{b}_N^0\|_N^2.$$

Next we take $\gamma = 1/2p^*\delta t$ and we define D_{p^*} as $\sup_{0\leq q\leq p^*}\|\mathbf{curl}\,\boldsymbol{b}_N^q\|_N^2$. We derive

$$\frac{1}{4\mu}D_{p^*} \leq \mu\delta t T\sum_{q=1}^{p^*-1}\|\frac{(\mathcal{I}_N\boldsymbol{j})((q+1)\delta t)-(\mathcal{I}_N\boldsymbol{j})((q-1)\delta t)}{\delta t}\|_{L^2(\Omega)^3}^2$$

$$+\,\frac{21\mu}{2}\sup_{0\leq q\leq p^*}\|(\mathcal{I}_N\boldsymbol{j})(q\delta t)\|_{L^2(\Omega)^3}^2 + 5\varepsilon\|\mathbf{curl}\,\boldsymbol{e}_{0N}\|_N^2 + \frac{7}{2\mu}\|\mathbf{curl}\,\boldsymbol{b}_N^0\|_N^2,$$

and similar estimates for the other terms. So the desired result follows from (3.10). □

The stability estimate (3.18) is rather complicated, since we take all the constants into account. However $\kappa_N^p(\boldsymbol{j})$ can be bounded by $\kappa(\mathcal{I}_N\boldsymbol{j})$ for the function κ introduced in (2.11), up to a multiplicative constant. And, if we introduce the function $\boldsymbol{b}_{N,\delta t}$, which is affine on each interval $[q\delta t,(q+1)\delta t]$ and equal to $\boldsymbol{b}_N^q$ in $q\delta t$, the left-hand side of (3.18) is the same as in (2.12), with $\boldsymbol{b}$ replaced by $\boldsymbol{b}_{N,\delta t}$. As usual, the estimate (3.18) will also be used to establish the error estimate.

4. Approximation results

We only prove the polynomial approximation results in the three-dimensional case, since they are much simpler in the two-dimensional one. We begin with the basic case where the domain Ω is a rectangular parallelepiped (which, in our case, would mean that Γ_c is empty, see Figure 1). Then we extend the results to the case where Ω is a union of rectangular parallelepipeds satisfying (3.1) and Assumption 3.1.

All the proofs in this section are derived from their finite element analogues; we refer to Nédélec [N1, §2] for the basic ideas, to Girault and Raviart [GR, Chap. III,

§5] and Amrouche, Bernardi, Dauge and Girault [ABDG, §4.a] for some extensions. Note that, as in [N1], we introduce a slightly smaller space than $V_N(\Omega)$ for proving the approximation properties. However, the choice of $V_N(\Omega)$ in the discrete problem leads to a simpler implementation than for this new space, since only one tensorized grid is unisolvent for all components of the elements of $V_N(\Omega)$.

Approximation in a rectangular parallelepiped. When Ω is a rectangular parallelepiped, we denote by $\mathcal{F}$ the set of its (six) faces and by $\mathcal{E}$ the set of its (twelve) edges. Also $\boldsymbol{\tau}_e$ stands for a unit vector tangential to any edge e. We use the notation $\mathbb{P}_{n_1,n_2,n_3}(K)$ for the space of restrictions to K of polynomials with degree $\leq n_j$ with respect to the j-th variable, $1 \leq j \leq 3$. And, for a positive integer N, we introduce the space

$$\mathbb{X}_N(\Omega) = \mathbb{P}_{N-1,N,N}(\Omega) \times \mathbb{P}_{N,N-1,N}(\Omega) \times \mathbb{P}_{N,N,N-1}(\Omega). \tag{4.1}$$

With this space, we associate [N1, Def. 6] the following operator r_N: for any sufficiently smooth function $\boldsymbol{v}$, $r_N\boldsymbol{v}$ belongs to $\mathbb{X}_N(\Omega)$ and satisfies

$$\begin{aligned}
&\forall e \in \mathcal{E}, \quad \forall q \in \mathbb{P}_{N-1}(e),\\
&\qquad\qquad \int_e (\boldsymbol{v} - r_N\boldsymbol{v}) \cdot \boldsymbol{\tau}_e\, q\, d\tau = 0,\\
&\forall f \in \mathcal{F}, \quad \forall \boldsymbol{q} \in \mathbb{P}_{N-2,N-1}(f) \times \mathbb{P}_{N-1,N-2}(f),\\
&\qquad\qquad \int_f (\boldsymbol{v} - r_N\boldsymbol{v}) \times \boldsymbol{n} \cdot \boldsymbol{q}\, d\boldsymbol{\tau} = 0,\\
&\forall \boldsymbol{q} \in \mathbb{P}_{N-1,N-2,N-2}(\Omega) \times \mathbb{P}_{N-2,N-1,N-2}(\Omega) \times \mathbb{P}_{N-2,N-2,N-1}(\Omega),\\
&\qquad\qquad \int_\Omega (\boldsymbol{v} - r_N\boldsymbol{v}) \cdot \boldsymbol{q}\, d\boldsymbol{x} = 0.
\end{aligned} \tag{4.2}$$

The operator r_N is uniquely defined by (4.2) [N1, Thm. 5]. Moreover, by exactly the same proof as in [ABDG, Lemma 4.7], it can be checked that, for any $p > 2$, it is continuous on

$$V^{0,p}(\Omega) = \Big\{\boldsymbol{v} \in L^p(\Omega)^3;\ \mathbf{curl}\,\boldsymbol{v} \in L^p(\Omega)^3 \text{ and } \boldsymbol{v} \times \boldsymbol{n} \in L^p(\partial\Omega)^2\Big\},$$

and it is also continuous on all functions $\boldsymbol{v}$ in $H(\mathbf{curl}, \Omega)$ such that $\boldsymbol{v} \times \boldsymbol{n}$ vanishes on $\partial\Omega$.

The idea for choosing the space $\mathbb{X}_N(\Omega)$ rather than $\mathbb{P}_N(\Omega)^3$ for the first approximation result relies on the following lemma, which is proven in [N1, Prop. 4] in the case of a tetrahedral decomposition.

Lemma 4.1. *The curl operator maps $\mathbb{X}_N(\Omega)$ onto the space of divergence-free polynomials in*

$$\mathbb{Y}_N(\Omega) = \mathbb{P}_{N,N-1,N-1}(\Omega) \times \mathbb{P}_{N-1,N,N-1}(\Omega) \times \mathbb{P}_{N-1,N-1,N}(\Omega). \tag{4.3}$$

Thanks to this lemma, the approximation result by polynomials in $\mathbb{X}_N(\Omega)$ becomes an obvious consequence of the approximation of divergence-free functions by divergence-free polynomials in $\mathbb{Y}_N(\Omega)$. Best approximation by divergence-free polynomials in $\mathbb{P}_{N-1}(\Omega)^3$ (which is contained in $\mathbb{Y}_N(\Omega)$) is analyzed by Sacchi-Landriani and Vandeven [SlV, Thm. 2.5] for smooth functions in $H_0^1(\Omega)^3 \cap H^4(\Omega)^3$, however the same arguments imply analogous results for all functions in $H^4(\Omega)^3$. Next, it can be extended to all functions in $H^s(\Omega)^3$, $s > 0$, thanks to an interpolation result of Bernardi, Dauge and Maday [BDM, Thm. 3.2].

Theorem 4.2. *For any real number $s > 0$, there exists a positive constant c such that, for all functions $\boldsymbol{v}$ in $H(\mathbf{curl}, \Omega)$ such that $\mathbf{curl}\, \boldsymbol{v}$ belongs to $H^s(\Omega)^3$,*

$$\inf_{\boldsymbol{v}_N \in \mathbb{X}_N(\Omega)} \|\mathbf{curl}\, \boldsymbol{v} - \mathbf{curl}\, \boldsymbol{v}_N\|_{L^2(\Omega)^3} \le c\, N^{-s}\, \|\mathbf{curl}\, \boldsymbol{v}\|_{H^s(\Omega)^3}. \tag{4.4}$$

Of course, estimate (4.4) is still valid when $\mathbb{X}_N(\Omega)$ is replaced by any space containing it, for instance $\mathbb{P}_N(\Omega)^3$.

Remark. Similar arguments allow us to prove that (4.4) still holds when the function $\boldsymbol{v}$ and all approximation polynomials $\boldsymbol{v}_N$ have their tangential traces equal to zero on one or several faces of Ω.

Approximation related to a conforming decomposition. Here we suppose that Ω is a polyhedral domain in $\mathbb{R}^3$ such that there exists a finite number of (open) rectangular parallelepipeds Ω_k, $1 \le k \le K$, satisfying (3.1) and Assumption 3.1. The discrete space is then defined by

$$X_N(\Omega) = \Big\{ v_N \in H(\mathbf{curl}, \Omega);\ v_{N\,|\Omega_k} \in \mathbb{X}_N(\Omega_k), 1 \le k \le K \Big\}. \tag{4.5}$$

And we introduce the operator $\mathcal{R}_N$ acting on sufficiently smooth functions $\boldsymbol{v}$:

$$(\mathcal{R}_N \boldsymbol{v})_{|\Omega_k} = r_N^k \boldsymbol{v}, \quad 1 \le k \le K,$$

where of course r_N^k stands for the previous operator r_N defined in (4.2) and translated on the rectangular parallelepiped Ω_k. It is proved in [N1, Thm. 5] that, if $\boldsymbol{n}_k$ denotes the unit exterior normal vector to Ω_k,

$$(\mathcal{R}_N \boldsymbol{v})_{|\Omega_k} \times \boldsymbol{n}_k + (\mathcal{R}_N \boldsymbol{v})_{|\Omega_{k'}} \times \boldsymbol{n}_{k'} = \mathbf{0}, \quad \text{on} \quad \partial\Omega_k \cap \partial\Omega_{k'}, \qquad 1 \le k < k' \le K,$$

so that $\mathcal{R}_N$ maps sufficiently smooth functions into $X_N(\Omega)$. So, it remains to check the good approximation properties of the operator r_N^k.

To prove these approximation estimates, we work on the cube $\Sigma =]-1,1[^3$, on which we define the operator r_N by (4.2) (with Ω replaced by Σ). The idea is to prove that, for all functions $\boldsymbol{v}$ in $H(\mathbf{curl}, \Sigma)$ with $\mathbf{curl}\, \boldsymbol{v}$ in $H^s(\Sigma)^3$,

$$\|\mathbf{curl}\, \boldsymbol{v} - \mathbf{curl}\, (r_N \boldsymbol{v})\|_{L^2(\Sigma)^3} \le c\, N^{-s}\, \|\mathbf{curl}\, \boldsymbol{v}\|_{H^s(\Sigma)^3}; \tag{4.6}$$

however we are also interested in the estimate in $L^2(\Sigma)^3$ when $\boldsymbol{v}$ belongs to $H^r(\Sigma)^3$:

$$\|\boldsymbol{v} - r_N \boldsymbol{v}\|_{L^2(\Sigma)^3} \le c\, N^{-r}\, \|\boldsymbol{v}\|_{H^r(\Sigma)^3}. \tag{4.7}$$

We need several lemmas and as many corollaries.

Lemma 4.3. *For all $s \ge 1$, estimate (4.6) holds for all functions $\boldsymbol{v}$ in $H(\mathbf{curl}, \Sigma)$ with $\mathbf{curl}\, \boldsymbol{v}$ in $H^s(\Sigma)^3$ and null tangential traces on the boundary of Σ.*

Proof. Any function $\boldsymbol{v}$ satisfying the hypotheses of the lemma can be written

$$v_x(x,y,z) = (1-y^2)(1-z^2) \sum_{m=0}^{+\infty} \sum_{n=1}^{+\infty} \sum_{p=1}^{+\infty} a_x^{mnp}\, L_m(x) L_n'(y) L_p'(z),$$

$$v_y(x,y,z) = (1-x^2)(1-z^2) \sum_{m=1}^{+\infty} \sum_{n=0}^{+\infty} \sum_{p=1}^{+\infty} a_y^{mnp}\, L_m'(x) L_n(y) L_p'(z),$$

$$v_z(x,y,z) = (1-x^2)(1-y^2) \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} \sum_{p=0}^{+\infty} a_z^{mnp}\, L_m'(x) L_n'(y) L_p(z),$$

so that $\boldsymbol{w} = \mathbf{curl}\,\boldsymbol{v}$ can be written

$$w_x(x,y,z) = (1-x^2) \sum_{m=1}^{+\infty} \sum_{n=0}^{+\infty} \sum_{p=0}^{+\infty} b_x^{mnp}\, L'_m(x) L_n(y) L_p(z),$$

$$w_y(x,y,z) = (1-y^2) \sum_{m=0}^{+\infty} \sum_{n=1}^{+\infty} \sum_{p=0}^{+\infty} b_y^{mnp}\, L_m(x) L'_n(y) L_p(z),$$

$$w_z(x,y,z) = (1-z^2) \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \sum_{p=1}^{+\infty} b_z^{mnp}\, L_m(x) L_n(y) L'_p(z).$$

We now take $\boldsymbol{q}$ equal to $(L_m(x)L'_n(y)L'_p(z), 0, 0)$, $0 \le m \le N-1$, $1 \le n,p \le N-1$, in the third line of (4.2), and analogous choices for the second and third components, and we observe that $\boldsymbol{v} - r_N\boldsymbol{v}$ admits the same decomposition as $\boldsymbol{v}$ with all the coefficients a_x^{mnp}, a_y^{mnp} and a_z^{mnp} for $m,n,p \le N-1$ equal to 0. This yields

$$\begin{aligned}\mathbf{curl}\,\boldsymbol{v} - \mathbf{curl}\,(r_N\boldsymbol{v}) = \Big(&w_x - \pi_N^{1,0(x)} \circ \pi_{N-1}^{(y)} \circ \pi_{N-1}^{(z)} w_x,\\ &w_y - \pi_{N-1}^{(x)} \circ \pi_N^{1,0(y)} \circ \pi_{N-1}^{(z)} w_y,\\ &w_z - \pi_{N-1}^{(x)} \circ \pi_{N-1}^{(y)} \circ \pi_N^{1,0(z)} w_z\Big),\end{aligned}$$

where π_{N-1}, resp. $\pi_N^{1,0}$, denotes the orthogonal projection operator from $L^2(-1,1)$ onto $\mathbb{P}_{N-1}(-1,1)$, resp. from $H_0^1(-1,1)$ onto $\mathbb{P}_N(-1,1) \cap H_0^1(-1,1)$, and the exponent (x), (y) or (z) represents the direction in which the operator is applied. Thus standard tensorization arguments give the desired result. $\square$

Corollary 4.4. *For all $r \ge 2$, estimate* (4.7) *holds for all functions $\boldsymbol{v}$ in $H(\mathbf{curl},\Sigma)$ with $\boldsymbol{v}$ in $H^r(\Sigma)^3$ and null tangential traces on the boundary of Σ.*

Proof. Using the same decomposition for $\boldsymbol{v}$ as in the proof of the previous lemma, we observe that

$$\begin{aligned}\boldsymbol{v} - r_N\boldsymbol{v} = \Big(&v_x - \pi_{N-1}^{(x)} \circ \pi_N^{1,0(y)} \circ \pi_N^{1,0(z)} v_x,\\ &v_y - \pi_N^{1,0(x)} \circ \pi_{N-1}^{(y)} \circ \pi_N^{1,0(z)} v_y, v_z - \pi_N^{1,0(x)} \circ \pi_N^{1,0(y)} \circ \pi_{N-1}^{(y)} v_z\Big).\end{aligned}$$

The end of the proof is the same as before. $\square$

Lemma 4.5. *For all $s \ge \frac{3}{2}$, estmate* (4.6) *holds for all functions $\boldsymbol{v}$ in $H(\mathbf{curl},\Sigma)$ with $\mathbf{curl}\,\boldsymbol{v}$ in $H^s(\Sigma)^3$ and null tangential traces on the edges of Σ.*

Proof. The idea is to write

$$\boldsymbol{v} = \boldsymbol{v}_0 + \sum_{f\in\mathcal{F}} \boldsymbol{v}_f,$$

where $\mathcal{F}$ here denotes the set of faces of Σ. Each $\boldsymbol{v}_f$ is chosen analogously to the next one which corresponds to the face $x = -1$:

$$\begin{aligned}(v_f)_x(x,y,z) &= 0,\\ (v_f)_y(x,y,z) &= \frac{1-x}{2}\, v_y(-1,y,z),\\ (v_f)_z(x,y,z) &= \frac{1-x}{2}\, v_z(-1,y,z).\end{aligned}$$

It must be noted that, due to the vanishing property on the edges, each $\boldsymbol{v}_f$ has a null tangential trace on other faces f' in $\mathcal{F}$, $f' \neq f$, so that $\boldsymbol{v}_0$ has null tangential traces on the boundary of Σ. Next, still on the face $x=-1$, it follows from the definition of $\mathbb{X}_N(\Sigma)$ together with the vanishing condition on the edges that $v_y(-1,y,z)$ and $v_z(-1,y,z)$ can be written as

$$v_y(-1,y,z) = (1-z^2) \sum_{n=0}^{+\infty} \sum_{p=1}^{+\infty} c_y^{mnp} \, L_n(y) L_p'(z),$$

$$v_z(-1,y,z) = (1-y^2) \sum_{n=1}^{+\infty} \sum_{p=0}^{+\infty} c_z^{mnp} \, L_n'(y) L_p(z),$$

so that taking $\boldsymbol{q}$ successively equal to $(L_n(y)L_p'(z), 0)$ and $(0, L_n'(y)L_p(z))$ in the second part of (4.2) yields that $\boldsymbol{v}_f - r_N \boldsymbol{v}_f$ admits the same decomposition as $\boldsymbol{v}_f$ with the coefficients c_y^{np} and c_z^{np} equal to 0 for $n, p \leq N-1$. As a consequence, setting $\boldsymbol{w}_f = \mathbf{curl}\, \boldsymbol{v}_f$, we derive

$$\begin{aligned} \mathbf{curl}\, \boldsymbol{v}_f - \mathbf{curl}\, (r_N \boldsymbol{v}_f) = \Big(& (w_f)_x - \pi_{N-1}^{(y)} \circ \pi_{N-1}^{(z)} (w_f)_x, \\ & (w_f)_y - \pi_N^{1,0(y)} \circ \pi_{N-1}^{(z)} (w_f)_y, \\ & (w_f)_z - \pi_{N-1}^{(y)} \circ \pi_N^{1,0(z)} (w_f)_z \Big), \end{aligned}$$

from which it is readily checked that

$$\|\mathbf{curl}\, \boldsymbol{v}_f - \mathbf{curl}\, (r_N \boldsymbol{v}_f)\|_{L^2(\Sigma)^3} \leq c\, N^{-s} \, \|\mathbf{curl}\, \boldsymbol{v}_f\|_{H^s(\Sigma)^3}.$$

This is the right estimate for $\mathbf{curl}\, \boldsymbol{v}_f$, but to derive it for $\mathbf{curl}\, \boldsymbol{v}$, we need a further investigation. We now introduce the operator π_N^1 from $H^1(-1,1)$ onto $\mathbb{P}_N(-1,1)$, defined as follows:

$$\pi_N^1 \varphi = \pi_N^{1,0} \varphi^0 + \frac{1-x}{2}\, \varphi(-1) + \frac{1+x}{2}\, \varphi(1),$$
$$\text{with} \quad \varphi^0 = \varphi - \frac{1-x}{2}\, \varphi(-1) - \frac{1+x}{2}\, \varphi(1).$$

And we use a modified decomposition $\mathbf{curl}\, \boldsymbol{v} = \boldsymbol{w}_1 + \boldsymbol{w}_2 + \boldsymbol{w}_3$, where for instance

$$\begin{aligned} (w_1)_x &= (\mathbf{curl}\, \boldsymbol{v}_0)_x + (\mathbf{curl}\, \boldsymbol{v}_{f_{x-}})_x + (\mathbf{curl}\, \boldsymbol{v}_{f_{x+}})_x, \\ (w_2)_x &= (\mathbf{curl}\, \boldsymbol{v}_{f_{y-}})_x + (\mathbf{curl}\, \boldsymbol{v}_{f_{y+}})_x, \\ (w_3)_x &= (\mathbf{curl}\, \boldsymbol{v}_{f_{z-}})_x + (\mathbf{curl}\, \boldsymbol{v}_{f_{z+}})_x, \end{aligned}$$

$f_{t\pm}$ denoting the face $t = \pm 1$. Then, it can be seen that

$$\begin{aligned} \big(\mathbf{curl}\, \boldsymbol{v} - \mathbf{curl}\, (r_N \boldsymbol{v})\big)_x = & \Big((w_1)_x - \pi_N^{1(x)} \circ \pi_{N-1}^{(y)} \circ \pi_{N-1}^{(z)} (w_1)_x \Big) \\ & + \Big((w_2)_x - \pi_N^{1,0(x)} \circ \pi_{N-1}^{(z)} (w_2)_x \Big) + \Big((w_3)_x - \pi_N^{1,0(x)} \circ \pi_{N-1}^{(y)} (w_3)_x \Big). \end{aligned}$$

However, replacing $\pi_N^{1,0}$ by its extension π_N^1 and noting that π_{N-1} is equal to the identity operator on $\mathbb{P}_1(-1,1)$, we derive that

$$\big(\mathbf{curl}\, \boldsymbol{v} - \mathbf{curl}\, (r_N \boldsymbol{v})\big)_x = \Big((\mathbf{curl}\, \boldsymbol{v})_x - \pi_N^{1(x)} \circ \pi_{N-1}^{(y)} \circ \pi_{N-1}^{(z)} (\mathbf{curl}\, \boldsymbol{v})_x \Big).$$

So, by the same tensorization arguments as before, we obtain the desired result. $\square$

We omit the proof of the corollary, which relies on similar arguments as the previous one.

Corollary 4.6. *For all $r \geq 2$, estimate* (4.7) *holds for all functions $\boldsymbol{v}$ in $H(\mathbf{curl}, \Sigma)$ with $\boldsymbol{v}$ in $H^r(\Sigma)^3$ and null tangential traces on the edges of Σ.*

Lemma 4.7. *For all $s \geq \frac{3}{2}$, estimate* (4.6) *holds for all functions $\boldsymbol{v}$ in $H(\mathbf{curl}, \Sigma)$ with $\mathbf{curl}\,\boldsymbol{v}$ in $H^s(\Sigma)^3$.*

Proof. It is similar to the previous one, since the idea is to write

$$\boldsymbol{v} = \boldsymbol{v}_0 + \sum_{f \in \mathcal{F}} \boldsymbol{v}_f + \sum_{e \in \mathcal{E}} \boldsymbol{v}_e,$$

where $\mathcal{E}$ denotes the set of edges of Σ. It can be checked that the contribution of the $\boldsymbol{v}_e$ consists in replacing $\pi_N^{1,0}$ by π_N^1 in the previous approximations of $(w_2)_x$ and $(w_3)_x$. This proves the final estimate. □

Corollary 4.8. *For all $r \geq 2$, estimate* (4.7) *holds for all functions $\boldsymbol{v}$ in $H(\mathbf{curl}, \Sigma)$ with $\boldsymbol{v}$ in $H^r(\Sigma)^3$.*

So the conclusion follows from Lemma 4.7 and Corollary 4.8:

Theorem 4.9. *For any real number $s \geq \frac{3}{2}$ and $r \geq 2$, there exists a positive constant c such that:*

(i) For all functions $\boldsymbol{v}$ in $H(\mathbf{curl}, \Omega)$ such that $(\mathbf{curl}\,\boldsymbol{v})_{|\Omega_k}$ belongs to $H^s(\Omega_k)^3$, $1 \leq k \leq K$,

$$\|\mathbf{curl}\,\boldsymbol{v} - \mathbf{curl}\,(\mathcal{R}_N \boldsymbol{v})\|_{L^2(\Omega)^3} \leq c\, N^{-s} \sum_{k=1}^{K} \|\mathbf{curl}\,\boldsymbol{v}\|_{H^s(\Omega_k)^3}; \tag{4.8}$$

(ii) For all functions $\boldsymbol{v}$ in $H(\mathbf{curl}, \Omega)$ such that $\boldsymbol{v}_{|\Omega_k}$ belongs to $H^r(\Omega_k)^3$, $1 \leq k \leq K$,

$$\|\boldsymbol{v} - \mathcal{R}_N \boldsymbol{v}\|_{L^2(\Omega)^3} \leq c\, N^{-r} \sum_{k=1}^{K} \|\boldsymbol{v}\|_{H^r(\Omega_k)^3}. \tag{4.9}$$

In view of the discrete problem (3.13), estimates of the tangential trace in the $L^2(\Gamma_a)^2$ norm are also needed.

Theorem 4.10. *For any real number $t \geq 1$, there exists a positive constant c such that, for all functions $\boldsymbol{v}$ in $H(\mathbf{curl}, \Omega)$ such that $\boldsymbol{v} \times \boldsymbol{n}_{|\overline{\Omega}_k \cap \Gamma_a}$ belongs to $H^t(\overline{\Omega}_k \cap \Gamma_a)^2$, $1 \leq k \leq K$,*

$$\|(\boldsymbol{v} - \mathcal{R}_N \boldsymbol{v}) \times \boldsymbol{n}\|_{L^2(\Gamma_a)^2} \leq c\, N^{-t} \sum_{k=1}^{K} \|\boldsymbol{v} \times \boldsymbol{n}\|_{H^t(\overline{\Omega}_k \cap \Gamma_a)^2}. \tag{4.10}$$

Proof. Here also, we work on the square Σ and establish the estimate, for instance, on the face f with equation $x = -1$. Indeed, it follows from the proof of Lemma 4.5 extended to the case of nonzero tangential traces on the edges that

$$\begin{aligned}
(v - r_N v)_y(-1, y, z) &= (v_y - \pi_{N-1}^{(y)} \circ \pi_N^{1(z)} v_y)(-1, y, z), \\
(v - r_N v)_z(-1, y, z) &= (v_z - \pi_N^{1(y)} \circ \pi_{N-1}^{(z)} v_z)(-1, y, z),
\end{aligned}$$

which gives the estimate. □

Finally, we introduce the orthogonal projection operator Π_N^c from $H(\mathbf{curl}, \Omega)$ onto $V_N(\Omega)$. To state its approximations properties, we also need the space

$$H^s(\mathbf{curl}, \Omega_k) = \left\{ \boldsymbol{v} \in H^s(\Omega_k)^3;\ \mathbf{curl}\,\boldsymbol{v} \in H^s(\Omega_k)^3 \right\}.$$

Theorem 4.11. *For any real number $s \geq 0$, there exists a positive constant c such that, for all functions $\boldsymbol{v}$ in $H(\mathbf{curl}, \Omega)$ such that $\boldsymbol{v}_{|\Omega_k}$ belongs to $H^s(\mathbf{curl}, \Omega_k)$, $1 \leq k \leq K$,*

$$\|\boldsymbol{v} - \Pi_N^c \boldsymbol{v}\|_{H(\mathbf{curl},\Omega)} \leq c\, N^{-s} \sum_{k=1}^{K} \|\boldsymbol{v}\|_{H^s(\mathbf{curl},\Omega_k)}. \tag{4.11}$$

Proof. Estimate (4.11) is true for $s = 0$ from the definition of Π_N^c, and for $s \geq 2$ from the inequality

$$\|\boldsymbol{v} - \Pi_N^c \boldsymbol{v}\|_{H(\mathbf{curl},\Omega)} \leq \|\boldsymbol{v} - \mathcal{R}_N \boldsymbol{v}\|_{H(\mathbf{curl},\Omega)},$$

and the previous Theorem 4.9. So, the general result follows thanks to an interpolation argument: the interpolation properties of the spaces $H^s(\mathbf{curl}, \Omega_k)$ can easily be deduced from [BDM, Thm. 3.1], by writing

$$H^s(\mathbf{curl}, \Omega_k) = \Big\{(\boldsymbol{v}, \boldsymbol{w}) \in H^s(\Omega_k)^3 \times H^s(\Omega_k)^3;\, \boldsymbol{w} = \mathbf{curl}\, \boldsymbol{v} \;\text{ in } \Omega_k\Big\}.$$

□

Different degrees of polynomials on different elements can be used in order to take advantage of the local regularity (on each Ω_k) that is required in the previous theorems. The way to derive estimates that involve the local regularity of the function consists in subtracting from the nonconforming approximation operator $\mathcal{R}_N$ a lifting of the jump of the tangential traces on the interfaces. However we do not consider this extension here.

5. Error estimates

We are now interested in deriving error estimates for the solution $\boldsymbol{b}$ of problem (2.9)(2.10), more precisely between the sequence of its values at $p\,\delta t$ and the sequence $(\boldsymbol{b}_N^p)_{0 \leq p \leq P}$ defined by (3.12)(3.13). It should be noted that, in any norm, this error is bounded by the sum of the errors coming respectively from the space and time discretizations. The space error comes from polynomial approximation and numerical integration of the data; the time error is linked to the consistency of the scheme.

We must now define the norm for these estimates. In view of (2.12) and (3.18), for a sequence $(\boldsymbol{v}^p)_{0 \leq p \leq P}$, it reads

$$\begin{aligned}\|(\boldsymbol{v}^p)\|_p = \Big(\varepsilon \,\|\frac{\boldsymbol{v}^{p+1} - \boldsymbol{v}^p}{\delta t}\|^2_{L^2(\Omega)^3} + \frac{1}{\mu}\,\|\mathbf{curl}\, \boldsymbol{v}^p\|^2_{L^2(\Omega)^3} \\ + \delta t \sqrt{\frac{\varepsilon}{\mu}} \sum_{q=0}^{p} \|\frac{\boldsymbol{v}^{q+1} - \boldsymbol{v}^q}{\delta t} \times \boldsymbol{n}\|^2_{L^2(\Gamma_a)^2}\Big)^{\frac{1}{2}}.\end{aligned} \tag{5.1}$$

It is the discrete form of the norm of

$$\mathcal{C}^1(0, T; L^2(\Omega)^3) \cap \mathcal{C}^0(0, T; V(\Omega)) \cap H^1(0, T; L^2(\Gamma_a)^2).$$

Abstract estimates. As usual, the first idea consists in applying the time scheme to the sequence made of the $\boldsymbol{b}(p\,\delta t)$, $0 \le p \le P$. Thus, when subtracting problem (2.9) at time $t = p\,\delta t$, we obtain, for all $\boldsymbol{v}$ in $V(\Omega)$,

$$\begin{aligned}\varepsilon \int_\Omega \frac{\boldsymbol{b}\big((p+1)\delta t\big) - 2\boldsymbol{b}(p\delta t) + \boldsymbol{b}\big((p-1)\delta t\big)}{(\delta t)^2} \cdot \boldsymbol{v}\, d\boldsymbol{x} + \frac{1}{\mu} \int_\Omega \mathbf{curl}\, \boldsymbol{b}(p\delta t) \cdot \mathbf{curl}\, \boldsymbol{v}\, d\boldsymbol{x} \\ + \sqrt{\frac{\varepsilon}{\mu}} \int_{\Gamma_a} \big(\frac{\boldsymbol{b}\big((p+1)\delta t\big) - \boldsymbol{b}\big((p-1)\delta t\big)}{2\delta t}\big) \times \boldsymbol{n} \cdot (\boldsymbol{v} \times \boldsymbol{n})\, d\boldsymbol{\tau} \\ = \int_\Omega \boldsymbol{j}(p\delta t) \cdot \mathbf{curl}\, \boldsymbol{v}\, d\boldsymbol{x} + \int_\Omega \boldsymbol{E}_1^p \cdot \boldsymbol{v}\, d\boldsymbol{x} + \int_{\Gamma_a} \boldsymbol{E}_3^p \cdot (\boldsymbol{v} \times \boldsymbol{n})\, d\boldsymbol{\tau}\end{aligned}$$

with

$$\begin{aligned}\boldsymbol{E}_1^p &= \varepsilon \Big(\frac{\boldsymbol{b}\big((p+1)\delta t\big) - 2\boldsymbol{b}(p\delta t) + \boldsymbol{b}\big((p-1)\delta t\big)}{(\delta t)^2} - (\partial_t^2 \boldsymbol{b})(p\delta t) \Big), \\ \boldsymbol{E}_3^p &= \sqrt{\frac{\varepsilon}{\mu}} \Big(\frac{\boldsymbol{b}\big((p+1)\delta t\big) - \boldsymbol{b}\big((p-1)\delta t\big)}{2\delta t} - (\partial_t \boldsymbol{b})(p\delta t) \Big) \times \boldsymbol{n}.\end{aligned}$$

For the sake of simplicity, we denote by $F^p(\boldsymbol{b})$ the quantity in $V(\Omega)'$ defined by

$$\begin{aligned}\langle F^p(\boldsymbol{b}), \boldsymbol{v} \rangle = \varepsilon \int_\Omega & \frac{\boldsymbol{b}\big((p+1)\delta t\big) - 2\boldsymbol{b}(p\delta t) + \boldsymbol{b}\big((p-1)\delta t\big)}{(\delta t)^2} \cdot \boldsymbol{v}\, d\boldsymbol{x} \\ & + \frac{1}{\mu} \int_\Omega \mathbf{curl}\, \boldsymbol{b}(p\delta t) \cdot \mathbf{curl}\, \boldsymbol{v}\, d\boldsymbol{x} \\ & + \sqrt{\frac{\varepsilon}{\mu}} \int_{\Gamma_a} \big(\frac{\boldsymbol{b}\big((p+1)\delta t\big) - \boldsymbol{b}\big((p-1)\delta t\big)}{2\delta t}\big) \times \boldsymbol{n} \cdot (\boldsymbol{v} \times \boldsymbol{n})\, d\boldsymbol{\tau},\end{aligned}$$

with obvious notation for the duality pairing. So we have proven that

$$\langle F^p(\boldsymbol{b}), \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{j}(p\delta t) \cdot \mathbf{curl}\, \boldsymbol{v}\, d\boldsymbol{x} + \int_\Omega \boldsymbol{E}_1^p \cdot \boldsymbol{v}\, d\boldsymbol{x} + \int_{\Gamma_a} \boldsymbol{E}_3^p \cdot (\boldsymbol{v} \times \boldsymbol{n})\, d\boldsymbol{\tau}, \tag{5.2}$$

and each term $\boldsymbol{E}_i^p$ represents a part of the consistency error.

Similarly, with any sequence $(\boldsymbol{w}_N^p)_{0 \le p \le P}$ of $V_N(\Omega)$, we associate the polynomial quantity $F_N^p(\boldsymbol{w}_N^p)$ defined by

$$\begin{aligned}((F_N^p(\boldsymbol{w}_N^p), \boldsymbol{v}_N))_N = \varepsilon\, ((\frac{\boldsymbol{w}_N^{p+1} - 2\boldsymbol{w}_N^p + \boldsymbol{w}_N^{p-1}}{(\delta t)^2}, \boldsymbol{v}_N))_N + \frac{1}{\mu}\, ((\mathbf{curl}\, \boldsymbol{w}_N^p, \mathbf{curl}\, \boldsymbol{v}_N))_N \\ + \sqrt{\frac{\varepsilon}{\mu}} ((\frac{\boldsymbol{w}_N^{p+1} - \boldsymbol{w}_N^{p-1}}{2\delta t} \times \boldsymbol{n}, \boldsymbol{v}_N \times \boldsymbol{n}))_{N,\Gamma_a}.\end{aligned}$$

The idea consists now in computing $F_N^p\big(\boldsymbol{b}_N^p - \mathcal{R}_{N-1}\boldsymbol{b}(p\,\delta t)\big)$. Indeed, since $\mathcal{R}_{N-1}\boldsymbol{b}$ belongs to $V_{N-1}(\Omega)$, we derive from the exactness of the quadrature formula and problem (3.13) that

$$((F_N^p\big(\boldsymbol{b}_N^p - \mathcal{R}_{N-1}\boldsymbol{b}(p\,\delta t)\big), \boldsymbol{v}_N))_N = \big((\boldsymbol{j}(p\delta t), \mathbf{curl}\, \boldsymbol{v}_N)\big)_N - \langle F^p(\mathcal{R}_{N-1}\boldsymbol{b}), \boldsymbol{v}_N \rangle.$$

Then adding (5.2) to this equality leads to

$$(5.3)\quad ((F_N^p(\boldsymbol{b}_N^p - \mathcal{R}_{N-1}\boldsymbol{b}(p\,\delta t)), \boldsymbol{v}_N))_N$$
$$= -\int_\Omega \boldsymbol{j}(p\delta t)\cdot \mathbf{curl}\,\boldsymbol{v}_N\,d\boldsymbol{x} + ((\boldsymbol{j}(p\delta t), \mathbf{curl}\,\boldsymbol{v}_N))_N$$
$$+ \langle F^p(\boldsymbol{b} - \mathcal{R}_{N-1}\boldsymbol{b}), \boldsymbol{v}_N\rangle - \int_\Omega \dot{\boldsymbol{E}}_1^p \cdot \boldsymbol{v}_N\,d\boldsymbol{x} - \int_{\Gamma_a} \boldsymbol{E}_3^p\cdot(\boldsymbol{v}_N\times\boldsymbol{n})\,d\boldsymbol{\tau}.$$

The first term in the right-hand side comes from numerical integration, while the second represents the approximation error.

Stability estimates. Next, we write each $\boldsymbol{b}_N^p - \mathcal{R}_{N-1}\boldsymbol{b}(p\,\delta t)$ as the sum $\boldsymbol{\eta}_0^p + \boldsymbol{\eta}_1^p + \boldsymbol{\eta}_2^p + \boldsymbol{\eta}_3^p$, where:

(i) The sequence $(\boldsymbol{\eta}_0^p)_{0\le p\le P}$ is the solution of problem (3.13) with null right-hand side and initial conditions

$$\boldsymbol{\eta}_0^0 = \boldsymbol{b}_{0N} - \mathcal{R}_{N-1}\boldsymbol{b}_0,$$
$$(5.4)\qquad \boldsymbol{\eta}_0^1 = \boldsymbol{b}_{0N} - \mathcal{R}_{N-1}\boldsymbol{b}_0 - \delta t\,\mathbf{curl}\,\boldsymbol{e}_{0N} - \int_0^{\delta t} (\partial_t\mathcal{R}_{N-1}\boldsymbol{b})(s)\,ds.$$

(ii) Each sequence $(\boldsymbol{\eta}_i^p)_{0\le p\le P}$, $i = 1, 2, 3$, has null initial values $\boldsymbol{\eta}_i^0 = \boldsymbol{\eta}_i^1 = \boldsymbol{0}$ and satisfies the following problem:

$$\forall \boldsymbol{v}_N \in V_N(\Omega),\quad ((F_N^{p+1}(\boldsymbol{\eta}_i^p), \boldsymbol{v}_N))_N = g_i^p,$$

where the quantities g_i^p are given by

$$g_1^p = -\int_\Omega [\mathbb{I} - \mathcal{R}_{N-1}]\Big(\frac{\boldsymbol{b}((p+1)\delta t) - 2\boldsymbol{b}(p\delta t) + \boldsymbol{b}((p-1)\delta t)}{(\delta t)^2}\Big)\cdot \boldsymbol{v}_N\,d\boldsymbol{x}$$
$$- \int_\Omega \boldsymbol{E}_1^p\cdot\boldsymbol{v}_N\,d\boldsymbol{x},$$
$$g_2^p = -\int_\Omega \mathbf{curl}\,\big([\mathbb{I} - \mathcal{R}_{N-1}]\boldsymbol{b}(p\delta t)\big)\cdot\mathbf{curl}\,\boldsymbol{v}_N\,d\boldsymbol{x}$$
$$- \int_\Omega \boldsymbol{j}(p\delta t)\cdot\mathbf{curl}\,\boldsymbol{v}_N\,d\boldsymbol{x} + ((\boldsymbol{j}(p\delta t), \mathbf{curl}\,\boldsymbol{v}_N))_N,$$
$$g_3^p = -\sqrt{\frac{\varepsilon}{\mu}}\int_{\Gamma_a} [\mathbb{I} - \mathcal{R}_{N-1}]\Big(\frac{\boldsymbol{b}((p+1)\delta t) - \boldsymbol{b}((p-1)\delta t)}{2\delta t}\Big)\times\boldsymbol{n}\cdot(\boldsymbol{v}_N\times\boldsymbol{n})\,d\boldsymbol{\tau}$$
$$- \int_{\Gamma_a} \boldsymbol{E}_3^p\cdot(\boldsymbol{v}_N\times\boldsymbol{n})\,d\boldsymbol{\tau}.$$

With each g_i^p, we associate a function $\boldsymbol{G}_i^p$ such that

$$g_1^p = \int_\Omega \boldsymbol{G}_1^p\cdot\boldsymbol{v}_N\,d\boldsymbol{x},\quad g_2^p = \int_\Omega \boldsymbol{G}_2^p\cdot\mathbf{curl}\,\boldsymbol{v}_N\,d\boldsymbol{x},\quad g_3^p = \int_{\Gamma_a} \boldsymbol{G}_3^p\cdot(\boldsymbol{v}_N\times\boldsymbol{n})\,d\boldsymbol{\tau}.$$

Indeed, applying the stability estimate (3.18) to the sequence $(\boldsymbol{\eta}_0^p)_{0\le p\le P}$ would yield

$$\|(\boldsymbol{\eta}_0^p)\|_p \le c\left(\|\mathbf{curl}\,\boldsymbol{\eta}_0^0\|_{L^2(\Omega)^3} + \left\|\frac{\boldsymbol{\eta}_0^1 - \boldsymbol{\eta}_0^0}{\delta t}\right\|_{L^2(\Omega)^3}\right);$$

however, a closer look at the proof of Proposition 3.3 leads to

$$\|(\boldsymbol{\eta}_0^p)\|_p \leq c(\|\mathbf{curl}\,\boldsymbol{\eta}_0^0\|_{L^2(\Omega)^3} + \|\mathbf{curl}\,(\boldsymbol{\eta}_0^1 - \boldsymbol{\eta}_0^0)\|_{L^2(\Omega)^3}). \tag{5.5}$$

Applying this same estimate (3.18) to the sequence $(\boldsymbol{\eta}_2^p)_{0\leq p\leq P}$ gives

$$\|(\boldsymbol{\eta}_2^p)\|_p^2 \leq c\,\tilde{\kappa}_N^p(\boldsymbol{G}_2), \tag{5.6}$$

where $\tilde{\kappa}_N^p(\boldsymbol{G}_2^p)$ is now defined by

$$\tilde{\kappa}_N^p(\boldsymbol{G}_2^p) = \mu\Big(\delta t T \sum_{q=0}^{p-1} \|\frac{\boldsymbol{G}_2^{q+1} - \boldsymbol{G}_2^{q-1}}{\delta t}\|_{L^2(\Omega)^3}^2 + 4 \sup_{0\leq q\leq p} \|\boldsymbol{G}_2^q\|_{L^2(\Omega)^3}^2\Big).$$

We must now prove similar stability properties for the sequences $(\boldsymbol{\eta}_1^p)_{0\leq p\leq P}$ and $(\boldsymbol{\eta}_3^p)_{0\leq p\leq P}$; however, they are solutions of simpler problems. The desired estimates are stated in the following lemmas.

Lemma 5.1. *If the parameters N and δt satisfy* (3.17), *the following estimate holds for the sequence $(\boldsymbol{\eta}_1^p)_{0\leq p\leq P}$:*

$$\|(\boldsymbol{\eta}_1^p)\|_p^2 \leq \frac{c}{\varepsilon}\delta t T \sum_{q=1}^{p} \|\boldsymbol{G}_1^q\|_{L^2(\Omega)^3}^2. \tag{5.7}$$

Proof. It is of course similar to that of Proposition 3.3. As for (3.19), taking $\boldsymbol{v}_N$ equal to $\boldsymbol{\eta}_1^{p+1} - \boldsymbol{\eta}_1^{p-1}$ yields

$$\varepsilon\,\|\frac{\boldsymbol{\eta}_1^{p+1} - \boldsymbol{\eta}_1^p}{\delta t}\|_N^2 + \frac{2}{3\mu}\,\|\mathbf{curl}\,\boldsymbol{\eta}_1^{p+1}\|_N^2 + 2\delta t\sqrt{\frac{\varepsilon}{\mu}}\sum_{q=1}^{p}\|\frac{\boldsymbol{\eta}_1^{q+1} - \boldsymbol{\eta}_1^{q-1}}{2\delta t} \times \boldsymbol{n}\|_{N,\Gamma_a}^2$$
$$\leq \frac{3}{4\mu}\,\|\mathbf{curl}\,(\boldsymbol{\eta}_1^{p+1} - \boldsymbol{\eta}_1^p)\|_N^2 + \sum_{q=1}^{p}\int_\Omega \boldsymbol{G}_1^q \cdot (\boldsymbol{\eta}_1^{q+1} - \boldsymbol{\eta}_1^{q-1})\,d\boldsymbol{x}.$$

Using (3.17) and the inequality (for $\gamma > 0$)

$$\sum_{q=1}^{p}\int_\Omega \boldsymbol{G}_1^q \cdot (\boldsymbol{\eta}_1^{q+1} - \boldsymbol{\eta}_1^{q-1})\,d\boldsymbol{x}$$
$$\leq \frac{\varepsilon}{8}\,\|\frac{\boldsymbol{\eta}_1^{p+1} - \boldsymbol{\eta}_1^p}{\delta t}\|_N^2 + \frac{\varepsilon}{8}\,\delta t\sum_{q=0}^{p-1}\gamma\|\frac{\boldsymbol{\eta}_1^{q+1} - \boldsymbol{\eta}_1^q}{\delta t}\|_N^2 + \frac{c}{\varepsilon}\,\delta t\sum_{q=1}^{p}\frac{1}{\gamma}\|\boldsymbol{G}_1^q\|_{L^2(\Omega)^3}^2,$$

we derive

$$\frac{\varepsilon}{8}\,\|\frac{\boldsymbol{\eta}_1^{p+1} - \boldsymbol{\eta}_1^p}{\delta t}\|_N^2 + \frac{2}{3\mu}\,\|\mathbf{curl}\,\boldsymbol{\eta}_1^{p+1}\|_N^2 + 2\delta t\sqrt{\frac{\varepsilon}{\mu}}\sum_{q=1}^{p}\|\frac{\boldsymbol{\eta}_1^{q+1} - \boldsymbol{\eta}_1^{q-1}}{2\delta t} \times \boldsymbol{n}\|_{N,\Gamma_a}^2$$
$$\leq \frac{\varepsilon}{8}\,\delta t\sum_{q=0}^{p-1}\gamma\|\frac{\boldsymbol{\eta}_1^{q+1} - \boldsymbol{\eta}_1^q}{\delta t}\|_N^2 + \frac{c}{\varepsilon}\,\delta t\sum_{q=1}^{p}\frac{1}{\gamma}\|\boldsymbol{G}_1^q\|_{L^2(\Omega)^3}^2.$$

Thus, the desired estimate follows from an appropriate choice of γ. □

We skip over the proof of the second lemma, since it is very similar to the previous one.

Lemma 5.2. *If the parameters N and δt satisfy* (3.17)*, the following estimate holds for the sequence* $(\boldsymbol{\eta}_3^p)_{0\le p\le P}$*:*

$$\|(\boldsymbol{\eta}_3^p)\|_p^2 \le c\sqrt{\frac{\mu}{\varepsilon}}\,\delta t T \sum_{q=1}^{p} \|\boldsymbol{G}_3^q\|^2_{L^2(\Gamma_a)^2}. \tag{5.8}$$

Conclusions. Estimating the right-hand members in (5.5)–(5.8) relies on the approximation properties of Section 4 for the approximation error terms, on Taylor's expansion for the consistency error terms, and on the standard interpolation error estimate [BM, Thm. 14.2] for the error due to numerical integration. This allows us to derive the estimate on $\|\boldsymbol{b}_N^p - \mathcal{R}_{N-1}\boldsymbol{b}(p\,\delta t)\|_p$ and then the estimate on $\|(\boldsymbol{b}(p\,\delta t) - \boldsymbol{b}_N^p)\|_p$ by a triangular inequality. We omit these technical details, and only state the final result.

Theorem 5.3. *Let s be a real number > 2. We assume that, for $1 \le k \le K$,*
(i) the function $\boldsymbol{j}$ satisfies

$$\boldsymbol{j}_{|\Omega_k} \in H^1(0,T;H^t(\Omega_k)^3), \quad t \ge s; \tag{5.9}$$

(ii) the solution $\boldsymbol{b}$ of problem (2.9)(2.10) *satisfies*

$$\begin{gathered}\boldsymbol{b}_{|\Omega_k} \in H^2(0,T;H^s(\Omega_k)^3), \quad \mathbf{curl}\,\boldsymbol{b}_{|\Omega_k} \in H^1(0,T;H^s(\Omega_k)^3),\\ (\boldsymbol{b}\times\boldsymbol{n})_{|\overline{\Omega}_k\cap\Gamma_a} \in H^1(0,T;H^s(\overline{\Omega}_k\cap\Gamma_a)^2),\end{gathered} \tag{5.10}$$

and also

$$\boldsymbol{b} \in \mathcal{C}^4(0,T;L^2(\Omega)^3), \quad \boldsymbol{b}\times\boldsymbol{n} \in \mathcal{C}^3(0,T;L^2(\Gamma_a)^2). \tag{5.11}$$

If, moreover, the parameters N and δt satisfy (3.17) *and the initial data are chosen such that*

$$\boldsymbol{b}_{0N} = \mathcal{I}_N\boldsymbol{b}_0 \quad \text{and} \quad \boldsymbol{e}_{0N} = \mathcal{I}_N\boldsymbol{e}_0, \tag{5.12}$$

the following error estimate holds between this solution and the sequence $(\boldsymbol{b}_N^p)_{0\le p\le P}$ defined in (3.12)(3.13)*, for $1 \le p \le P-1$:*

(5.13)

$$\|(\boldsymbol{b}(p\,\delta t) - \boldsymbol{b}_N^p)\|_p \le cT\Big(N^{-s}\sum_{k=1}^{K}(\|\boldsymbol{j}\|_{H^1(0,T;H^s(\Omega_k)^3)} + \|\boldsymbol{b}\|_{sp,k}) + T(\delta t)^2\,\|\boldsymbol{b}\|_{ti}\Big),$$

where $\|\cdot\|_{sp,k}$, resp. $\|\cdot\|_{ti}$, denotes the norm of the space in (5.10)*, resp.* (5.11)*.*

Estimate (5.13) is optimal in the sense that the required regularity properties of the solution are minimal with respect to the order of convergence. Moreover, the space regularity that appears in the estimate is local on each Ω_k. So, using different degrees of polynomials on the subdomains would allow for a better convergence. The corresponding analysis is presently under consideration.

The discretization is of order two with respect to the time variable and of spectral type with respect to the space variable: the exponent of N in (5.13) depends only on the regularity of the data and the exact solution. Since δt has to be chosen very small in order for (3.17) to hold, the global convergence rate seems very good: it behaves like $c\,N^{-4}$ for sufficiently smooth solutions.

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